J. Fluid Mech. (1988), *vol.* 193, *pp.* 75–110
Printed in Great Britain

A new theory of the instability of a uniform fluidized bed

By G. K. BATCHELOR
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

(Received 25 August 1987 and in revised form 10 February 1988)

The form of the momentum equation for one-dimensional (vertical) unsteady mean motion of solid particles in a fluidized bed or a sedimenting dispersion is established from physical arguments. In the case of a fluidized bed that is slightly non-uniform this equation contains two dependent variables, the local mean particle velocity V and the local concentration ϕ, and several statistical parameters of the particle motion in a uniform bed. All these parameters are functions of ϕ with clear physical meanings, and the important ones are measurable. It is a novel feature of the equation that it contains two explicit contributions to the bulk modulus of elasticity of the particle configuration, one arising from the transfer of particle momentum by velocity fluctuations and one arising from the effective repulsive force exerted between particles in random motion. This latter contribution, which proves to be the more important of the two, is related to the gradient diffusivity of the particles, a key quantity in the new theory.

The equation of mean motion of the particles and the equation of particle conservation are sufficient to determine the behaviour of a small disturbance with sinusoidal variation of V and ϕ in the vertical direction. Particle inertia forces in such a propagating wavy disturbance may promote amplitude growth, whereas particle diffusion tends to suppress it, and instability occurs when the particle Froude number exceeds a critical value. Rough estimates of the relevant parameters allow the criterion for instability to be put in approximate numerical form for both gas-fluidized beds (for which the flow Reynolds number at marginal stability is small) and liquid-fluidized beds of solid spherical particles (for which the Reynolds number is well above unity), although more information about the particle diffusivity in particular is needed. The predictions of the theory appear to be in qualitative accord with the available observational data on instability of gas- and liquid-fluidized beds.

---

1. Introduction

Stationary solid particles in an open vertical cylinder which are supported by a plate spanning the cylinder form a 'packed bed'. If the plate is porous and gas or liquid is forced through the plate from below, the bed remains packed until the flow rate reaches a certain value. At this critical value the bed expands a little and the particles become mobile or 'fluidized'. Further increase of the flow rate to a new steady value causes the bed to expand further. Under certain conditions this further expansion is uniform, and the bed remains statistically homogeneous, with the number density of the particles taking just the value required for the weight of a particle to be balanced by the mean drag force exerted by the fluid. Chemical engineers use such a fluidized bed as a means of obtaining a high rate of heat or mass transfer between the particles and the fluid and between the particles and sidewalls.

Particles dispersed throughout the fluid in a stationary closed vertical cylinder likewise fall vertically relative to the fluid under the action of gravity, and a dispersion of identical particles that is initially statistically homogeneous usually remains homogeneous during sedimentation, except at the top and bottom of the cloud. It is observed that the steady mean fall speed of the particles is a decreasing function of the particle concentration, corresponding to the expansion of a uniform fluidized bed with increase of the flow rate. This 'hindered setting' is a consequence of hydrodynamic interference of the particles; the more crowded they are, the larger is the drag coefficient of each particle.

A fluidized bed of particles is simply a cloud of sedimenting particles referred to different axes, and one would expect the theoretical and experimental investigations of the two situations to have proceeded together. In fact, they have developed separately, presumably because the different technologies associated with fluidized beds in chemical engineering and with sedimentation processes in colloid science and mining and reservoir engineering are associated with different parameter ranges. Fluidized beds have mostly been studied with larger particles, such that the Reynolds number of the flow about a particle is well above unity, whereas sedimentation processes in practice usually involve a liquid continuous phase and smaller particles for which the Reynolds number is small. The subject matter of this paper concerns a range of particle sizes for which the particle Reynolds numbers are not all either large or small compared with unity, and concepts and results developed in the two fields will be used.

It is a common and practically important feature of gas-fluidized beds that, at flow rates above a certain value, 'bubbles' in which the particle concentration is evidently quite small form and rise to the surface of the bed. For fine particles of powder fluidized in air at normal pressure the minimum flow rate for the occurrence of bubbles is found to be clearly above the minimum flow rate for fluidization (Geldart 1973), whereas for solid particles with diameters above about 200 μm fluidized in air, bubbles appear at flow rates close to the minimum for fluidization. Bubbles of relatively clear fluid also appear to form spontaneously in liquid-fluidized beds, although often only for flow rates well above the minimum for fluidization and for particles of large size or density (Davidson & Harrison 1963).

There is an enormous literature recording investigations of the occurrence, the properties, and the effects of bubbles in fluidized beds, but their origin is not well understood. A natural and common speculation is that bubbles originate in some instability of a statistically uniform fluidized bed which causes growth of the amplitude of fluctuations in the particle concentration. A consideration of types of small disturbance to which a uniform fluidized bed might be unstable suggests a wavy disturbance with variation of the concentration in the vertical direction alone, since such a disturbance interacts with the mean motion of the particles as a consequence of the dependence of the mean velocity of particles on the local concentration.

It was pointed out by Kynch (1952) that, when the particle concentration is a sufficiently slowly varying function of the vertical position coordinate x (measured downwards), the mean vertical velocity of a sedimenting particle may be assumed to be determined by the local number density n and so to be approximately equal to the mean velocity in a uniform dispersion with the same concentration, $U(n)$ say. The differential equation expressing conservation of particles then becomes

$$\frac{\partial n}{\partial t}+\frac{\partial(nU)}{\partial x}=\frac{\partial n}{\partial t}+\frac{\mathrm{d}(nU)}{\mathrm{d}n}\frac{\partial n}{\partial x}=0, \tag{1.1}$$

showing that a small slowly varying departure from a uniform concentration propagates without change of form with the vertical velocity

$$U+n\frac{\mathrm{d}U}{\mathrm{d}n}; \tag{1.2}$$

and since $\mathrm{d}|U|/\mathrm{d}n<0$ this propagation velocity is upward relative to the particles. Kynch had in mind small sedimenting particles for which the flow Reynolds number is small, but the same remarks apply to any dispersion of identical particles regardless of their size. Now the effect of inertia of the particles, which is neglected in Kynch's analysis, will be to cause some delay in the adjustment of the mean particle velocity to a change in the local concentration, and therein lies the possibility that over one cycle of a periodic wave there might be an increase in the amplitude of oscillation of the particles.

The observation of bubbles in a fluidized bed thus provides some motivation for a consideration of the instability of a uniform fluidized bed, the expectation being that the vertical mean density gradients associated with a wavy disturbance of exponentially increasing amplitude might ultimately cause gravitational overturning on the scale of the wavelength, out of which bubbles of nearly clear fluid form in some way. That expectation has not yet been confirmed, either theoretically or experimentally. But regardless of the precise connection with bubbles, the instability of a uniform fluidized bed is a sufficiently interesting phenomenon in its own right to justify full investigation.

The specific purpose of this paper is to consider theoretically the effect of particle inertia forces and other consequences of non-negligible spatial gradients of concentration on vertically propagating concentration waves of small amplitude, and to look in particular for conditions under which the amplitude grows exponentially. This is not a new problem, and there are many published papers that purport to show theoretically the existence of growing waves. However, no rational theory in full accord with the known facts has yet been put forward; and even though there is a general belief that a uniform fluidized bed may be unstable, the underlying physical mechanisms are not yet clear. The shortcomings of the existing theories derive, I believe, from the unsatisfactory nature of the proposed models or equations from which the behaviour of a disturbance has been determined. An important preliminary purpose of the paper therefore is to establish carefully, with plausible physical reasoning and a minimum of hypothesis and model making, the form of the equations describing one-dimensional unsteady mean motion of the particles in a fluidized bed. Since the motion of a particle is random and averaging of dependent variables is essential, we cannot expect to be able to establish a closed finite set of equations that will be sufficient to determine the mean motion by calculation alone. Some hypotheses concerning the relation between mean quantities will be needed, as in the analogous problem of turbulence in pure fluid, but we shall try to avoid the introduction of any variables or material parameters that do not have a clear physical meaning and are not calculable or measurable, at least in principle.

Since we are proceeding *ab initio*, and are not building on previous theoretical developments, it is not necessary to describe published work at this stage. Reference will be made to observations and results obtained by other authors in §6.

An index of symbols specifying their meanings or the places in the text where they are defined may be found at the end of the paper.

2. The equations governing one-dimensional unsteady mean motion of the particles

In this section we consider the general form of the equations that govern the mean motion of the particles in the vertical direction, with questions about the values of parameters being left until later.

The fluidizing fluid under consideration may be either a liquid or a gas, with density ρ_f. The particles are assumed to be identical, and to have uniform density ρ_p and mass m, and to be large enough for effects of Brownian motion of the particles to be negligible. No specific assumption need be made initially about the nature of the particle material, although it is the practical case of solid particles falling relative to the fluid in a fluidized bed that will later be a basis for approximations and empirical formulae. Both fluidizing fluid and particles are assumed to be incompressible.

The fluidized bed is assumed to be statistically homogeneous in each horizontal plane. The mean particle velocity is (exactly) vertical, and is not necessarily steady. Mean values, to be denoted by angle brackets where a special symbol is not introduced, can be regarded as ensemble averages, that is, averages over a large number of realizations of the system with the same macroscopic external conditions. In view of the horizontal homogeneity an ensemble mean is equivalent to a spatial average over a horizontal plane. If we imagine two horizontal planes close together at vertical position x (measured downward), a count of the number of particle centres found instantaneously between two adjoining large areas of these planes gives the mean number density $n(x, t)$. 'Close together' here means that the separation is small compared with the distance over which the change in n is appreciable. And if the vertical component of velocity of each particle is recorded, that determines a mean particle velocity $V(x, t)$ and a fluctuation v about that mean with statistical properties such as $\langle v^2 \rangle$. V and v are signed quantities and are positive when directed downwards, like x and the gravitational acceleration g.

Now since the particles and fluid are incompressible, the flux of material volume across a large area of a horizontal plane is independent of x. We shall assume that this volume flux is also time-independent, that is, that there is no acceleration of the mixture as a whole. The axes of reference adopted here are such that the mean flux of material volume across a horizontal plane is zero. These axes of reference are more natural for observations of sedimentation of particles in a stationary container than for a fluidized bed of particles, but we shall nevertheless refer to the particles as a fluidized bed for lack of an alternative collective noun. I hope that those who are more familiar with the literature of fluidized beds than that of sedimentation will have no difficulty in adapting their thinking to these axes, which seem to be more appropriate for the basic dynamics. The results are of course unaffected since these are not accelerating axes.

The most significant measure of the local concentration of the particles is the particle volume fraction $\phi(x, t)$, where

$$\phi \rho_p = nm. \tag{2.1}$$

The 'voidage' fraction $1-\phi$ is often introduced as an alternative to ϕ. In view of the choice of reference frame the mean flux of fluid volume across a horizontal plane surface is $-\phi V$, and the mean vertical component of the velocity in the fluid is $-\phi V/(1-\phi)$.

In a statistically homogeneous fluidized bed the mean particle velocity is independent of x and t, and is a function of ϕ and of the particle properties, to be denoted as $U(\phi)$. This function of ϕ is known empirically for solid spherical particles.

Each of the governing equations to be established expresses conservation of some quantity, particle number or particle momentum. By considering first the rate of change of number of particle centres located within a fixed cylindrical domain with vertical generators and horizontal plane end faces of large area at arbitrary positions we find

$$\frac{\partial n}{\partial t}+\frac{\partial(nV)}{\partial x} = 0. \tag{2.2}$$

When the spatial gradients of concentration are so small that the mean particle velocity depends only on the local concentration and in the same way as in a homogeneous bed, so that $V \approx U(\phi)$, then (2.2) is sufficient for the determination of the history of small disturbances, as Kynch (1952) showed. But in more general circumstances the effect of particle inertia is significant and a second equation is needed to match the two independent variables ϕ and V.

The momentum equation is best approached formally in order to minimize the risk of overlooking some relevant physical process. We consider the balance of particle momentum in the same cylindrical control volume, of volume τ and area A of each of the end faces, and begin with the verbal statement:

rate of change of mean momentum of particles with centres in τ
= mean flux of particle momentum inward across the boundary of τ
+ mean force exerted on particles in τ by gravity (or any other externally imposed body force)
+ mean force exerted on particles in τ by fluid
+ mean force exerted on particles in τ by particles outside τ. (2.3)

The mean momentum of the particles per unit volume of the mixture is nmV, so the term on the left-hand side is

$$Am\int_{x_1}^{x_2}\frac{\partial(nV)}{\partial t}\,\mathrm{d}x. \tag{2.4}$$

where $x = x_1$ and $x = x_2$ are the positions of the end faces of the cylindrical control volume.

Of the four contributions on the right-hand side of (2.3), the first and fourth represent transfers of momentum or forces of limited range acting across the surface bounding the control volume and are proportional to A (the contribution from the transfer across the curved surface of the cylinder being zero from the symmetry about the vertical), and the second and third represent forces acting on particles throughout the cylinder and are expressed as integrals over the volume τ. We consider each of these four terms in turn.

(1) The mean flux of particle momentum across a horizontal plane surface of area

A due to particles crossing that surface is $Amn(V^2+\langle v^2\rangle)$, so the first term on the right-hand side of (2.3) is

$$-Am\int_{x_1}^{x_2}\frac{\partial\{n(V^2+\langle v^2\rangle)\}}{\partial x}\,\mathrm{d}x. \tag{2.5}$$

The contribution from fluctuations in the velocity of a particle is analogous to the Reynolds stress in the mean momentum equation for turbulent flow of a fluid. These velocity fluctuations might arise from variations in the configuration of particles and the resulting hydrodynamic interactions, or, in the case of high-Reynolds-number flow around a particle, from turbulence in the fluid. $\langle v^2\rangle$ is measurable, at any rate in a homogeneous fluidized bed or sedimenting dispersion, and observations may be recorded in the literature although I have not been able to find any that would define $\langle v^2\rangle/U^2$ as a function of ϕ numerically.

(2) The mean total force exerted on the particles with centres instantaneously within τ by gravity, as modified by buoyancy due to the action of gravity on the fluid (which strictly speaking is a part of the third contribution to the right-hand side of (2.3) but it is convenient to transfer it to the second), is

$$Am\tilde{g}\int_{x_1}^{x_2} n\,\mathrm{d}x, \tag{2.6}$$

where $\tilde{g} = g(\rho_p-\rho_f)/\rho_p$. The expressions for the mean force on the particles due to other externally imposed fields penetrating into the interior of the control volume (electrostatic, magnetic, centrifugal) may likewise be constructed.

(3) One can recognize three physically different types of contribution to the mean force exerted on a particle by the fluid. The first and most obvious type of contribution is associated with frictional resistance and dissipation of energy in the fluid, and if the dispersion is homogenous this is the only contribution. We introduce the function $F_h(V,\phi)$ representing the mean force exerted by the fluid on a particle whose mean velocity is V in a homogenous dispersion of concentration ϕ. For the purpose of definition of this function the independent variable V may take any value (implying different possible values of the externally applied force). We know that $V = U$ when gravity drives the motion, so that

$$F_h(U,\phi) = -m\tilde{g}. \tag{2.7}$$

I do not think the dissipative contribution at a point in a non-uniform dispersion where the mean particle velocity is V and the concentration is ϕ will be significantly different from $F_h(V,\phi)$ unless the spatial gradients are large, although I cannot justify this. (It is reassuring that later we shall be concerned with the behaviour of *small* disturbances to a homogenous fluidized bed.) Thus we assume the dissipative contribution may be represented as $F_h(V,\phi)$ to a reasonable approximation.

The other two types of contribution are associated with non-zero values of $\partial V/\partial t$, $\partial V/\partial x$, $\partial\phi/\partial t$ or $\partial\phi/\partial x$; one concerns the direct effect of inertia of the fluid (an indirect effect of fluid inertia is of course present in the dissipative contribution when the particle Reynolds number is not small) and the other a transport effect arising from the random fluctuations in the velocity of a particle. In the absence of a systematic procedure we can have no guarantee that there are not more than two and that no important physical process has been overlooked. However, overlooking a contribution can be remedied later, and so seems preferable to the unwitting inclusion of a spurious contribution required by a hypothetical model.

Regarding the contribution due to fluid inertia, it is clear that a change in the mean velocity of a particle generates momentum of the fluid and is accompanied by an 'acceleration reaction' on the particle. In the case of an isolated body moving in infinite inviscid fluid this acceleration reaction is known to be equivalent to the addition of a virtual mass to the real mass of the body, and the value of the virtual mass can be found from irrotational-flow theory to be of the form $C_0 m\rho_f/\rho_p$, where C_0 is a constant of order unity. There are two difficulties to be overcome in a generalization of this classical theory to the case of acceleration of solid particles dispersed in real fluid. The first, which is more fundamental and more severe, is that the viscous fluid through which the solid particles are moving cannot realistically be supposed to be free from vorticity at any Reynolds number. If the acceleration of the particles is of large magnitude, the resulting *change* in the fluid velocity is approximately irrotational, and it might be possible to use the classical theory to determine the fluid reaction on a particle during the period of large acceleration. However, this would not be of much help to us, because the particle accelerations in question here arise from the propagation of a wavy disturbance of small amplitude through the dispersion and so are of small magnitude. The second difficulty is that multiple hydrodynamic interaction of particles is important at the non-small values of ϕ that are relevant to instability of a fluidized bed. Some progress has been made in the generalization of the classical theory to allow for pair interactions, but hydrodynamic interactions of more than two particles in irrotational flow are outside the present scope of calculations.

We do not know enough about the fluid flow due to accelerating solid particles to be able to represent the acceleration reaction analytically with any confidence. However it seems fairly certain that the order of magnitude of the ratio of the acceleration reaction to the particle inertia force is ρ_f/ρ_p, which fortunately is small for all gas-fluidized beds of solid particles and for some liquid-fluidized beds. I propose here to adopt a simple form for the mean fluid force on a particle resulting from change of the mean particle velocity V, viz.

$$-m\frac{\rho_f}{\rho_p}\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)(CV), \tag{2.8}$$

where C is a function of ϕ with magnitude of order unity. This has the right limiting form as $\phi \to 0, C \to C_0$ (when particles are effectively isolated and the fluid motion is irrotational outside a wake that is narrow at large Reynolds number) and can be regarded as a plausible generalization of the case of a dilute dispersion when the fluid flow is irrotational, as Biesheuval & van Wijngaarden (1984) have pointed out. More relevantly, I believe (2.8) probably represents correctly, in order of magnitude, the acceleration reaction on solid particles at arbitrary ϕ. It can thus serve as an indicator of the importance of the effects of inertia of the fluid on the force exerted on particles. If we find later that the force contribution (2.8) plays a non-negligible part in the conditions for instability of a fluidized bed, that will be a signal that the errors involved in the use of (2.8) are of some consequence and that there is need for further investigation of the effects of fluid inertia.

A transport process associated with a force exerted on a particle in a non-uniform fluidized bed is particle diffusion. The velocity of a particle in a fluidized bed or sedimenting dispersion fluctuates randomly owing to the continual change in the configuration of neighbouring particles and the resulting hydrodynamic interactions (and perhaps owing also to turbulence in the wake of large particles). This random velocity fluctuation gives the particle a statistical tendency to migrate, as in the case of a small colloidal particle in Brownian motion, and in the presence of a gradient of concentration of particles there will be a net flux of particle number down the gradient, $\mathscr{F}$ say. When the lengthscale of the random walk of the particles is small by comparison with the lengthscale on which $\partial\phi/\partial x$ varies, the statistical migration constitutes a diffusion process, with the flux of particle number being of the linear form $-D\,\partial n/\partial x$, where D is the local (hydrodynamic) particle diffusivity in the vertical direction. Now this particle flux is equivalent to an additional mean particle velocity $\mathscr{F}/n$, which can be regarded as a consequence of a steady force exerted on each particle. Furthermore, if we exclude for the moment the possibility that particles sometimes touch, this effective steady force is literally exerted by the fluid. I say 'effective', because the migration is statistical and the fluid force actually fluctuates in time and varies from one particle to another. However, the mean or effective force causing the diffusional flux is just what we want for the mean momentum equation (2.3). To obtain the mean force from the additional mean velocity that it produces, we divide by the bulk mobility of the particles, B say, defined as the ratio of the (small additional) mean velocity, relative to zero-volume-flux axes, to the (small additional) steady force applied to each particle of a homogeneous dispersion.† The bulk mobility B is clearly a function of ϕ.

There is thus a transport contribution $\mathscr{F}/nB$ to the mean force exerted on a particle by the fluid, and when the condition for the existence of a particle diffusion process is satisfied – as we shall henceforth assume to be so – this can be written as

$$\frac{\mathscr{F}}{nB} = -\frac{D}{nB}\frac{\partial n}{\partial x}. \tag{2.9}$$

Those readers who are not familiar with the association of the diffusion of particles with an effective force acting on them may be wondering how this force is manifested in reality. The subsequent discussion of the last term in (2.3), where the diffusion contribution (2.9) could also logically be included, bears on this question.

On gathering up the above three contributions to the mean force exerted on a particle by the fluid we have for the third term on the right-hand side of (2.3)

$$A\int_{x_1}^{x_2}\left\{nF_{\mathrm{f}}(V,\phi) - mn\theta\left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x}\right) + mn\zeta V\frac{\partial V}{\partial x} - \frac{D}{B}\frac{\partial n}{\partial x}\right\}\mathrm{d}x, \tag{2.10}$$

in which we have made use of (2.2) and written

$$\theta(\phi) = \frac{\rho_{\mathrm{f}}}{\rho_{\mathrm{p}}}C(\phi), \quad \zeta(\phi) = \frac{\rho_{\mathrm{f}}}{\rho_{\mathrm{p}}}\phi\frac{\mathrm{d}C}{\mathrm{d}\phi}. \tag{2.11}$$

(4) The nature of the fourth and last term on the right-hand side of (2.3) may be explained by reference to a hypothetical case in which the particles are electrically charged and exert repulsive electrostatic forces on each other. The range of action of these electrostatic forces is small by comparison with the dimensions of the dispersion, and so the mean resultant force exerted on the particles inside τ by those outside τ is represented by a force per unit area of the surface bounding of the volume τ, that is, by a stress, $-S$ say, which is a function of the local particle concentration.

† The words 'small additional' are needed when the relation between mean velocity and applied force is nonlinear.

Electrostatic interparticle forces are conservative, and in that case one can interpret $-S$ as the derivative of the mean potential energy per particle with respect to the volume of the mixture per particle. The contribution to the net force exerted on particles in our control volume by external particles is then

$$-A\int_{x_1}^{x_2}\frac{\partial S}{\partial x}\mathrm{d}x. \tag{2.12}$$

A repulsive force between particles corresponds to a positive value of S (relative to zero when the particles are far apart), in which case S plays a dynamical role analogous to the pressure in a gas.

There is also an effective force between particles exerted across the surface of the control volume which is associated with particle velocity fluctuations and encounters between moving particles. When a particle within the control volume approaches the boundary surface it may happen that it is also approaching a particle whose centre lies outside the control volume and close to the bounding surface. The presence of this outside particle is an impediment to the motion of the inside particle, and has an effect equivalent to the exertion of a (repulsive) force on the inside particle. In the case of solid particles dispersed in a gas, the particles may make touching collisions, in which case the force exerted on the inside particle by the outside particle during the small period of contact is determined by the elastic stress at the common surface of contact of the two particles. When the continuous phase is a liquid, actual contact between two particles normally does not occur, owing to the strong resistance of the intervening liquid to being squeezed out, but the outside particle is still exerting an effective force on the inside particle. Another way of looking at this interparticle force associated with velocity fluctuations and the exclusion of particle overlap is to say that it is the mean normal stress that must be exerted at the boundary of a homogeneous region containing many particles to prevent them from being dispersed by their velocity fluctuations. (It is helpful to regard the boundary as moving with the mean speed of the particles.)

There is a concept in statistical mechanics, viz. the chemical potential, which has some relevance here. The chemical potential of small particles in Brownian motion in stationary fluid is a measure of the free energy associated with their concentration, and two dispersions in separate containers put side by side will be in thermodynamic equilibrium (meaning that the separating wall could be removed without causing any change) only if the chemical energy per particle (μ, say), as well as the temperature and fluid pressure, is the same on the two sides. Furthermore, if a gradient of chemical potential is set up in a tube connecting the two containers, perhaps as a result of the concentrations in the two containers being different, the diffusive flux of number of particles down the gradient (relative to zero-volume-flux axes) is the same as if each particle is acted on by a steady force equal to $-(1-\phi)^{-1}\nabla\mu$ (Batchelor 1976). Thus the contribution to the stress function S that would be made by small Brownian non-sedimenting particles is given by

$$-\frac{\partial S}{\partial x}=-\frac{n}{1-\phi}\frac{\partial\mu}{\partial x}=-\frac{D}{B}\frac{\partial n}{\partial x}, \tag{2.13}$$

where $D(\phi)$ is the (Brownian) diffusivity of the particles and $B(\phi)$ is the bulk mobility defined as before. An expression for μ in terms of the interparticle force law is available, and in particular μ can be calculated as a function of ϕ for particles that exert an actual force on each other only when they are touching and are resisting an attempt to make them overlap.

The relation (2.13) is exact for very small particles whose velocity fluctuations are due to their share of the thermal energy of the system. As already mentioned, the larger sedimenting particles with which we are concerned likewise exhibit random velocity fluctuations, due to the continual change in the configuration of particles and the resulting hydrodynamic interactions. There is no precise definition of the chemical potential in the non-Brownian system, but all the other quantities in (2.13) have definite mechanical meanings (D being now the *hydrodynamic* diffusivity) and the equality of the first and last members holds provided the concentration gradient is constant over a sufficiently large range of values of x for the relation between flux of particles and concentration gradient to be linear. This equality simply says that the addition to the mean velocity of the particles associated with diffusion down a concentration gradient is a consequence of the non-uniformity of an effective repulsive force between particles represented by the stress function S. Thus we have recovered the transport contribution (2.9) from a different starting point.

It appears then that the contribution to the momentum equation due to particle diffusion cannot be said unequivocally to fall in either the third or the fourth term on the right-hand side of (2.3). If particles never touch, the force in question is literally exerted by fluid stresses at the surface of a particle, but the origin of the force lies in the presence and motion of other particles and it corresponds more to the physical reality to say that the force is exerted by other particles. However, the proper classification of the force is a secondary matter, because the equality of the first and last members of (2.13) holds in all circumstances.

Finally, we note that the particle stress $-S$ may be non-zero when the mean particle velocity is non-uniform, even in the absence of fluctuations in particle velocity. When the mean rate of strain $\partial V/\partial x$ is negative particles are coming closer together, and the viscous fluid resists being squeezed from between particles. This resistance is equivalent to the exertion of a repulsive force between particles (and actually is such a force when two particles touch). Provided inertia forces are negligible in the small gaps between particles and the mean velocity gradient is uniform over distances of a few particle diameters, the particle stress will be linear in $\partial V/\partial x$ and vanishes with ϕ, in which case the contribution to S may be written as

$$S = -\,\phi\rho_{\mathrm{f}}\,\eta'\frac{\partial V}{\partial x}. \tag{2.14}$$

The parameter $\phi\rho_{\mathrm{f}}\,\eta'$ here is a positive function of ϕ with the dimensions of viscosity, and represents a resistance to deformation of the configuration of particles which is likely to be important when particles are very close, as for example in a slowly contracting layer of small particles that have fallen through liquid to the bottom of a container.

The two contributions to the mean total force on particles within the control volume exerted by particles outside the control volume that have been identified here are first that due to particle diffusion, represented by (2.13), and second that due to resistance to configurational deformation, represented by (2.14). The former has already been included in (2.10), so that the fourth term on the right-hand side of (2.3) consists of (2.12) with the expression (2.14) for S.

On inserting in (2.3) the analytical expressions (2.4), (2.5), (2.6), (2.10) and (2.12) with (2.14), and using the fact that the momentum balance holds for all choices of x_1 and x_2, we obtain a differential equation which becomes

$$mn(1+\theta)\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)-mn\zeta V\frac{\partial V}{\partial x} = -\frac{\partial(mn\langle v^2\rangle)}{\partial x}+n\{F_{\mathrm{h}}(V,\phi)-F_{\mathrm{h}}(U,\phi)\}$$
$$-\frac{D}{B}\frac{\partial n}{\partial x}+\frac{\partial}{\partial x}\left(\phi\rho_{\mathrm{f}}\eta'\frac{\partial V}{\partial x}\right) \quad (2.15)$$

when use is made of (2.7) and the particle-conservation equation (2.2). The form of the acceleration-reaction term is a provisional guess, but otherwise no strong approximation or assumption or restriction has been made in obtaining the two conservation equations (2.2) and (2.15) for one-dimensional mean motion, and all quantities in these equations have clear physical meanings. If enough information about the various parameters in (2.15) is available, these two equations are sufficient in principle for the determination of V and ϕ as functions of x and t. Note that there is no need, in this problem of one-dimensional mean motion of incompressible phases, to consider the mean momentum balance for the fluid because the mean fluid velocity is determined by V and ϕ. As we shall see, not all the parameters in (2.15) have a significant influence on the stability of a uniform fluidized bed. One parameter that does prove to be important is the particle diffusivity D. The presence of this measurable parameter in the equation of mean motion of the particles is the main novel feature of the present theory.

3. The form of the equations for small departures from uniformity

In preparation for the analysis of the stability of a homogeneous fluidized bed to small disturbances, we consider now the approximate form taken by the momentum equation (2.15) when the departures from homogeneity are small and the spatial gradients $\partial\phi/\partial x$ and $\partial V/\partial x$ are small in some sense. This requires separate consideration of the quantities F_{h} and $\langle v^2\rangle$ on the right-hand side of (2.15). No further consideration of the form of other terms in (2.15) is needed, but we note that the implicit assumption underlying some of them – that the statistical properties of the particle configuration other than ϕ are approximately constant – becomes more accurate as the departures from homogeneity decrease.

The mean force exerted by the fluid on a particle with mean speed V in a homogeneous bed of concentration ϕ may be written as

$$F_{\mathrm{h}}(V) = -\tfrac{1}{2}\pi a^2\rho_{\mathrm{f}}V|V|C_D, \quad (3.1)$$

where a is the volume-equivalent radius of the particle, and the drag coefficient C_D depends on the flow Reynolds number

$$\mathbb{R} = 2a|V|\rho_{\mathrm{f}}/\mu \quad (3.2)$$

as well as on ϕ and possibly on the density ratio $\rho_{\mathrm{p}}/\rho_{\mathrm{f}}$. The statistical configuration of the particles is also relevant but is not an independent variable in a homogeneous dispersion of particles. In the case of fine particles for which $\mathbb{R} \ll 1$ the flow is dominated by viscous forces and $C_D \propto \mathbb{R}^{-1}$, whereas when $\mathbb{R} \gg 1$ the effects of fluid inertia are dominant (for solid particles) over most of the flow field. Since at this stage we are unsure about the particle size that may be critical for instability, it is desirable to impose no restrictions on the value of $\mathbb{R}$. However we can make use of the fact that C_D varies slowly with $\mathbb{R}$, and that the number

$$\gamma = \frac{V}{F_h}\left(\frac{\partial F_h}{\partial V}\right)_{\phi\,\mathrm{const}} \tag{3.3}$$

varies slowly, between 1 when $\mathbb{R} \ll 1$ and a maximum of 2 when $\mathbb{R} \gg 1$. Since γ is approximately constant over a small range of mean particle speeds, we may regard it as being evaluated at the particular value $V = U$. It is believed (Wallis 1969) that the expression (3.3) also does not vary much with ϕ, so the value of γ can presumably be estimated from the known dependence of the drag coefficient on Reynolds number for an isolated particle.

Now in a perturbed fluidized bed the particle volume fraction ϕ and mean particle velocity V vary slightly with x and t, and V differs slightly from the gravitational velocity U appropriate to the local value of ϕ. If we again make the reasonable assumption that the statistical configuration of particles does not vary significantly in the disturbed fluidized bed, it follows that the first approximation to the fluctuation in the local dissipative fluid force term in (2.15) is given by

$$F_h(V) - F_h(U) = \gamma\left(\frac{V-U}{U}\right)F_h(U) = -\gamma m\tilde{g}\left(\frac{V-U}{U}\right). \tag{3.4}$$

The mean-square velocity fluctuation in a homogeneous fluidized bed is a function of the particle volume fraction ϕ, and no doubt also of the flow Reynolds number and the density ratio ρ_p/ρ_f, which we write as

$$\langle v^2\rangle_h = H(\phi)\,U^2. \tag{3.5}$$

There are no velocity fluctuations in the absence of hydrodynamic interactions between particles, so $H \to 0$ as $\phi \to 0$. At the other extreme, when ϕ approaches the close-packing limit, $\langle v^2\rangle_h$ must again approach zero, whereas although U^2 becomes small it remains non-zero in the limit. It seems therefore that $H(\phi)$ has a maximum at some intermediate value of ϕ.

Now in the presence of small gradients of mean particle speed and number density the value of $\langle v^2\rangle$ will differ from (3.5) as a consequence of the bias in the velocities of the particles arriving at station x from the two different directions, just as the value of the mean-square velocity of molecules of a gas is changed slightly by the existence of a gradient of mean molecular velocity. For a small departure from the homogeneous state we may appeal to phenomenological arguments and write

$$\langle v^2\rangle = H(\phi)\,U^2 - \eta''(\phi)\frac{\partial V}{\partial x} - \eta'''(\phi)\,U\frac{\partial \phi}{\partial x}, \tag{3.6}$$

where the coefficients η'' and η''' have the dimensions of a diffusivity and are parameters of the homogeneous state depending primarily on ϕ and secondarily on $\mathbb{R}$ and ρ_p/ρ_f. A positive gradient of V should cause a decrease in the value of $\langle v^2\rangle$ for the same reasons as in the kinetic theory of gases, and as defined in (3.6) η'' should therefore be positive. η'' represents the diffusion of particle momentum, and $mn\eta''$ is a kind of viscosity. The significance of η''' is less evident. Arguments like those used in kinetic theory of gases suggest that $|\eta'''| \ll \eta''$. Both gradient terms in (3.6) play the secondary role of damping disturbances with small lengthscale in the later considerations of stability of a fluidized bed. We shall therefore drop the term containing η''' in the absence of evidence to suggest it should be retained.

It will be observed that we now have two terms on the right-hand side of (2.15) involving resistance to a rate of deformation of the configuration of particles, viz.

$$\frac{\partial}{\partial x}\left(mn\eta''\frac{\partial V}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\phi\rho_{\rm f}\,\eta'\frac{\partial V}{\partial x}\right), \tag{3.7}$$

which can conveniently be lumped together by putting

$$\rho_{\rm p}\,\eta''+\rho_{\rm f}\,\eta' = \rho_{\rm p}\,\eta. \tag{3.8}$$

The new parameter $\phi\rho_{\rm p}\,\eta$ may be termed the particle viscosity and represents the sum of two different effects, one being the diffusion of particle momentum by the random fluctuations in particle velocity and the other resistance to change of the particle spacing due to viscous stresses in the fluid.

With regard to the other terms in (2.15), we need only note that the bulk mobility B introduced in (2.9) is defined as $-\{\partial F_{\rm h}(V)/\partial V\}^{-1}_{V=U}$ and so may be written with the aid of (3.3) as

$$B = -\frac{U}{\gamma F_{\rm h}(U)} = \frac{U(\phi)}{\gamma m\tilde{g}}. \tag{3.9}$$

γ is effectively a constant for a perturbed bed, so B is a known function of ϕ.

The form of the momentum equation (2.15) that is appropriate to a perturbed fluidized bed in which spatial gradients are small is now found by substituting (3.4), (3.6) (with $\eta''' = 0$), (3.8) and (3.9) in the terms on the right-hand side. Thus

$$\phi(1+\theta)\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)-\phi\zeta V\frac{\partial V}{\partial x} = -\frac{\mathrm{d}(\phi HU^2)}{\mathrm{d}\phi}\frac{\partial\phi}{\partial x}+\frac{\partial}{\partial x}\left(\phi\eta\frac{\partial V}{\partial x}\right)-\frac{\gamma\tilde{g}}{U}\left\{\phi(V-U)+D\frac{\partial\phi}{\partial x}\right\}. \tag{3.10}$$

It is of interest at this point to notice the correspondence between (3.10) and the mean momentum equation for a flowing gas of density mn (or $\phi\rho_{\rm p}$) composed of discrete molecules. Aside from the fluid drag-weight term (3.4) and the acceleration-reaction terms, to which there are no counterparts in the gas-flow equation since the molecules move in a vacuum, the two equations are formally identical, as might have been expected. The quantity $-mn\langle v^2\rangle$ in (2.15), supplemented by (3.6), corresponds to the mean normal stress in an ideal gas of molecules of negligible volume which exert no force on each other except when in collision; part of this normal stress (viz. $\phi\rho_{\rm p}HU^2$) is a 'pressure' dependent on the local density and part ($\rho_{\rm p}\,\phi\eta''\,\partial V/\partial x$) is a 'deviatoric stress' proportional to the local mean rate of strain. The quantity $-S$ introduced in (2.12) represents an additional stress due to particle interactions, and likewise contains both a pressure part (involving D) and a deviatoric part (involving η'). The effects of particle interactions are normally small for a flowing gas (in which case they are called 'real gas' effects) but may be expected to play a more important role in two-phase flow in view of the strength of the hydrodynamic interactions between adjacent moving particles. The total coefficient of $-\partial\phi/\partial x$ on the right-hand side of (3.10), viz.

$$\frac{\mathrm{d}(\phi HU^2)}{\mathrm{d}\phi}+\frac{\gamma\tilde{g}D}{U}, \quad = Q \text{ say}, \tag{3.11}$$

is a parameter of a homogeneous fluidized bed which is primarily a function of ϕ, and $\phi\rho_{\rm p}\,Q$ may be interpreted as a bulk modulus of elasticity of the configuration of particles.

It could be said that the derivation of (3.10) given here justifies an assumption that the particle phase in a fluidized bed behaves like a gas occupying the whole space, with the addition of terms representing the force exerted on the particles in unit volume by the second phase. However, the advantages in deriving the particle momentum equation in the above manner instead of simply postulating a momentum equation of the general form appropriate to a gas in motion are first that we know precisely *how* general must be the assumed form of the momentum equation for the gas (for instance, would one have known that 'real gas' effects should be allowed for?) and second that each term of (3.10) can be interpreted in terms of mechanical processes involving the actual particles.

We now suppose that a homogeneous fluidized bed characterized by the particle volume fraction ϕ_1 is disturbed and that in the disturbed state

$$\phi = \phi_1 + \phi', \quad V = U_1 + V', \tag{3.12}$$

where the suffix 1, here and in the two later equations, denotes the value of a function of ϕ at $\phi = \phi_1$. All terms of the two governing equations (2.2) and (3.10) vanish in the case of the homogeneous bed, and the equations for the small perturbation quantities ϕ' and V', correct to first order, are as follows:

$$\frac{\partial \phi'}{\partial t} + U_1 \frac{\partial \phi'}{\partial x} + \phi_1 \frac{\partial V'}{\partial x} = 0, \tag{3.13}$$

$$\phi_1(1+\theta_1)\left(\frac{\partial V'}{\partial t} + U_1 \frac{\partial V'}{\partial x}\right) - \phi_1 \zeta_1 U_1 \frac{\partial V'}{\partial x}$$
$$= -K_1 \frac{\partial \phi'}{\partial x} + \phi_1 \eta_1 \frac{\partial^2 V'}{\partial x^2} - \frac{\gamma \tilde{g}}{U_1}\left(\phi_1 V' - W_1 \phi' + D_1 \frac{\partial \phi'}{\partial x}\right), \tag{3.14}$$

in which we have written

$$\left\{\frac{\mathrm{d}(\phi H U^2)}{\mathrm{d}\phi}\right\}_{\phi_1} = K_1, \quad \left(\phi \frac{\mathrm{d}U}{\mathrm{d}\phi}\right)_{\phi_1} = W_1. \tag{3.15}$$

Since all quantities in (3.13) and (3.14) other than x and t and the disturbance magnitudes ϕ' and V' are parameters of the undisturbed homogeneous fluidized fluid, the suffix 1 is redundant and will be dropped.

We get some insight into the relative magnitudes of the different terms in (3.14) by making the variables non-dimensional. Suppose that a disturbance with lengthscale λ is imposed on the bed. It is likely that velocities will scale with U and times with λ/U. We therefore put

$$X = \frac{x}{\lambda}, \quad T = \frac{tU}{\lambda},$$

and rewrite (3.14) as

$$\phi(1+\theta)\left(\frac{\partial}{\partial T} + \frac{\partial}{\partial X}\right)\frac{V'}{U} - \phi\zeta \frac{\partial(V'/U)}{\partial X}$$
$$= -\frac{K}{U^2}\frac{\partial \phi'}{\partial X} + \frac{\phi\eta}{\lambda U}\frac{\partial^2(V'/U)}{\partial X^2} - \frac{\gamma \tilde{g} \lambda}{U^2}\left(\frac{\phi V'}{U} - \frac{W\phi'}{U} + \frac{D}{\lambda U}\frac{\partial \phi'}{\partial X}\right). \tag{3.16}$$

All terms in (3.16) other than that multiplied by $\gamma\tilde{g}\lambda/U^2$ (and the normally negligible part of the particle viscosity coming from $\phi\rho_f\eta'$) represent inertial effects of different kinds. It follows that, as

$$U^2/\gamma|\tilde{g}|\lambda \to 0$$

and all other non-dimensional parameters in (3.16) remain unchanged, inertial effects become relatively small and that (on reverting to dimensional variables)

$$\phi V' \approx -\phi' W - D\frac{\partial \phi'}{\partial x}, \tag{3.17}$$

whence ϕ' is seen from (3.13) to be given by

$$\frac{\partial \phi'}{\partial t} + (U+W)\frac{\partial \phi'}{\partial x} = D\frac{\partial^2 \phi'}{\partial x^2}. \tag{3.18}$$

This is the familiar kinematic-wave equation for ϕ' in a sedimenting dispersion with very small gradients (Kynch 1952) supplemented by a new diffusion or damping term, and there is a solution representing a sinusoidal wave of wavelength λ propagating with velocity W relative to the particles and with amplitude proportional to $\exp(-4\pi^2 Dt/\lambda^2)$.

It is interesting to note that the diffusion term has its familiar place in (3.18), and it might be supposed that a diffusion term should have been included on the right-hand side of the particle-conservation equation from the beginning. I think that would have been a mistake in the context of consideration of effects of particle inertia. All contributions to the mean flux of particles, including that from diffusion down a concentration gradient, are contributions also to the mean particle momentum, and so the mean velocity that is the dependent variable in the momentum balance equation should include the contribution from diffusion; diffusion is then incorporated in the particle flux term in the conservation equation (2.2). Actually the consequences of including the diffusion term explicitly in the particle-conservation equation instead of in the momentum equation are not of major importance for the problem of stability, and in particular the criterion for growth of a disturbance to be given later is unaffected in the absence of acceleration-reaction effects.

When on the other hand

$$U^2/\gamma|\bar{g}|\lambda \to \infty$$

and all other non-dimensional parameters remain unchanged, the non-inertial terms in (3.16) become negligible and we have

$$(1+\theta)\left(\frac{\partial V'}{\partial t} + U\frac{\partial V'}{\partial x}\right) - \zeta U\frac{\partial V'}{\partial x} = -\frac{K}{\phi}\frac{\partial \phi'}{\partial x} + \eta\frac{\partial^2 V'}{\partial x^2}. \tag{3.19}$$

Equations (3.13) and (3.19) are identical, aside from the extra term involving ζ in (3.19), with those describing small disturbances to the density and velocity of a compressible viscous gas in uniform motion with velocity U. By eliminating ϕ' from (3.13) and (3.19) it may be seen that a sinusoidal disturbance propagates vertically with no change of form other than a slow decrease in amplitude due wholly to the particle viscosity. The two possible wave velocities are rather complicated functions of U, ϕ, K, θ, ζ and η, but if the minor influence of the particle viscosity on these wave velocities is ignored they become

$$U\left(1 - \frac{\frac{1}{2}\zeta}{1+\theta}\right) \mp \left\{\frac{K}{1+\theta} + \frac{\frac{1}{4}\zeta^2 U^2}{(1+\theta)^2}\right\}^{\frac{1}{2}}, \tag{3.20}$$

provided of course that the quantity within curly brackets is positive. These dynamic waves involving inertia forces and effective elasticity of the particle configuration may propagate with the same speed, relative to axes moving with velocity $U\{1-\frac{1}{2}\zeta(1+\theta)^{-1}\}$, in either of the two directions and are mechanically similar to compression waves in a gas, although the random velocity fluctuations that provide the resistance to compression are of thermal origin in the case of the gas molecules.

Regarding the sign of the square-rooted expression in (3.20), there is no doubt that K (defined in (3.15)) is positive at small values of ϕ but may be negative at some large values of ϕ, as we shall see. A negative value of this expression would imply exponential growth of a non-propagating disturbance and the spontaneous formation of regions of high concentration of particles. We need more data on $\langle v^2\rangle$ as a function of ϕ for a homogeneous fluidized bed before we can make predictions about this kind of instability at large values of $U^2/\gamma|\tilde{g}|\,\lambda$.

The intermediate regime in which inertia forces and variations of the fluid drag force are both significant is evidently a bridge between a regime in which damped kinematic waves may propagate and one in which damped dynamic waves may propagate. If unstable wavy disturbances exist in a fluidized bed, they presumably occur in this intermediate regime which we now examine.

4. The behaviour of sinusoidal disturbances to a homogeneous fluidized bed

An initial small disturbance to a homogeneous fluidized bed may be resolved into Fourier components which evolve independently. We therefore consider a disturbance that varies sinusoidally with respect to x with wavenumber $\kappa(=2\pi/\lambda)$. The governing equations (3.13) and (3.14) are linear and homogeneous with constant coefficients, so an exponential dependence on t may be anticipated. We put

$$\phi' = A\,\mathrm{e}^{\mathrm{i}\kappa(x-ct)}, \quad V' = B\,\mathrm{e}^{\mathrm{i}\kappa(x-ct)}, \tag{4.1}$$

where A and B are constants and κ is real but c may be complex. Substitution in (3.13) and (3.14) (and suppression of the suffix 1) gives

$$A(U-c)+B\phi = 0, \tag{4.2}$$

$$B\phi(1+\theta)(U-c)-B\phi\zeta U = -AK+\mathrm{i}B\phi\kappa\eta+\frac{\gamma\tilde{g}}{\kappa U}(\mathrm{i}B\phi-\mathrm{i}AW-A\kappa D). \tag{4.3}$$

The condition for the existence of a non-zero solution of (4.2) and (4.3) for A and B is

$$(1+\theta)(c-U)^2+(c-U)\left(\zeta U+\mathrm{i}\kappa\eta+\frac{\mathrm{i}\gamma\tilde{g}}{\kappa U}\right)-\left\{K+\frac{\gamma\tilde{g}}{\kappa U}(\mathrm{i}W+\kappa D)\right\}=0.$$

Before solving this quadratic equation for c it pays to rewrite it as

$$(c-\hat{U})^2+i\hat{P}(c-\hat{U})-\hat{Q}-\frac{\mathrm{i}\gamma\tilde{g}\hat{W}}{\kappa U}=0, \tag{4.4}$$

where the following substitutions have been made:

$$\left.\begin{aligned}\hat{P}&=\frac{\kappa^2\eta U+\gamma\tilde{g}}{\kappa U(1+\theta)}, & \hat{Q}&=\frac{KU+\gamma\tilde{g}D}{U(1+\theta)}+\frac{\frac{1}{4}\zeta^2U^2}{(1+\theta)^2},\\ \hat{U}&=U\left(1-\frac{\frac{1}{2}\zeta}{1+\theta}\right), & \hat{W}&=\frac{W}{1+\theta}+\frac{\frac{1}{2}\zeta U(\kappa^2\eta U+\gamma\tilde{g})}{\gamma\tilde{g}(1+\theta)^2}.\end{aligned}\right\} \tag{4.5}$$

Note that in the absence of acceleration-reaction effects (i.e. when $\phi = \zeta = 0$), the quantities denoted by $\hat{Q}$, $\hat{U}$ and $\hat{W}$ reduce to Q (see (3.11)), U and W. The solution of (4.4) for c is

$$c = \hat{U} - \tfrac{1}{2}\mathrm{i}\hat{P} \mp \left(\tfrac{1}{4}\hat{P}^2 - \hat{Q} - \frac{\mathrm{i}\gamma\tilde{g}\hat{W}}{\kappa U}\right)^{\frac{1}{2}}. \tag{4.6}$$

Since the real and imaginary parts of the complex number $(a+\mathrm{i}b)^{\frac{1}{2}}$, where a and b are real, are

$$\{\tfrac{1}{2}a + \tfrac{1}{2}(a^2+b^2)^{\frac{1}{2}}\}^{\frac{1}{2}} \quad \text{and} \quad \{-\tfrac{1}{2}a + \tfrac{1}{2}(a^2+b^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$$

respectively, we see from (4.6) that the real and imaginary parts of c are

$$c_{\mathrm{r}} = \hat{U} \pm \left[-\tfrac{1}{8}\hat{P}^2 + \tfrac{1}{2}\hat{Q} + \tfrac{1}{2}\left\{(\tfrac{1}{4}\hat{P}^2 - \hat{Q})^2 + \left(\frac{\gamma\tilde{g}\hat{W}}{\kappa U}\right)^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}, \tag{4.7}$$

$$c_{\mathrm{i}} = -\tfrac{1}{2}\hat{P} \mp \left[\tfrac{1}{8}\hat{P}^2 - \tfrac{1}{2}\hat{Q} + \tfrac{1}{2}\left\{(\tfrac{1}{4}\hat{P}^2 - \hat{Q})^2 + \left(\frac{\gamma\tilde{g}\hat{W}}{\kappa U}\right)^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{4.8}$$

The condition for disturbances with exponentially growing amplitude to exist, that is, for $c_{\mathrm{i}} > 0$, becomes evident when the inner square-rooted quantity in (4.8) is rewritten as

$$(\tfrac{1}{4}\hat{P}^2 + \hat{Q})^2 + \left(\frac{\gamma\tilde{g}\hat{W}}{\kappa U}\right)^2 - \hat{P}^2\hat{Q};$$

it is

$$(\gamma\tilde{g}\hat{W})^2 > \kappa^2U^2\hat{P}^2\hat{Q} \tag{4.9}$$

If $\hat{Q} < 0$ this condition is always satisfied, as one would expect for particles with a negative bulk modulus of elasticity ($Q < 0$ being a necessary condition for $\hat{Q} < 0$). However, the estimates of K and D to be given in the next section suggest that the more realistic and interesting case is $\hat{Q} > 0$, and the condition (4.9) is then equivalent to

$$N = \frac{(\gamma\tilde{g}\hat{W})^2}{\kappa^2U^2\hat{P}^2\hat{Q}} = \frac{\{(1+\theta)\gamma\tilde{g}W + \tfrac{1}{2}\zeta U(\kappa^2\eta U + \gamma\tilde{g})\}^2}{(\kappa^2\eta U + \gamma\tilde{g})^2\{(1+\theta)Q + \tfrac{1}{4}\zeta^2U^2\}} > 1. \tag{4.10}$$

A connection may now be made with the results obtained in the preceding section for small and large values of $U^2/\gamma|\tilde{g}|\lambda$. The non-dimensionalizing length λ was not specified in §3, but we can identify it here with the disturbance wavelength $2\pi/\kappa$. There is actually another length occurring implicitly in equation (3.10), viz. the particle dimension a, which is expected to be the length factor relevant to the two 'diffusivities' D and η. It was required there that the order of magnitude of dimensionless quantities such as κa should not change during the two limiting operations that led to (3.17) and (3.19). If now we put $\kappa U^2/\gamma|\tilde{g}| \ll 1$ *and* $U^2/\gamma|\tilde{g}|a \ll 1$ in (4.10), the terms proportional to $\gamma|\tilde{g}|$ in the expressions for $\hat{P}$, $\hat{Q}$ and $\hat{W}$ are dominant and the condition (4.10) for growing waves to exist is not satisfied. One of the two solutions in (4.7) and (4.8) then represents a wave propagating with velocity $U+W$ (the kinematic-wave velocity) and amplitude diminishing as $\exp(-\kappa^2Dt)$; the other and less familiar solution retains some influence of inertia forces and represents a strongly damped wave propagating with velocity $U - W - \zeta(1+\theta)^{-1}U$. If on the other hand $\kappa U^2/\gamma|\tilde{g}| \gg 1$, N can be seen to be smaller than unity provided $Q > 0$, and the two solutions in (4.7) and (4.8) then represent two dynamic waves propagating with the velocities (3.20) and damping due to the particle viscosity, as found previously. We shall see however that N *does* exceed unity for certain small values of $\kappa U^2/\gamma|\tilde{g}|$ and order-one values of $U^2/\gamma|\tilde{g}|a$, a possibility that was not evident from the previous discussion of the two limiting cases and that shows instability when inertial effects are neither negligible nor dominant.

The dimensionless parameter N depends on the wavenumber of the disturbance and has a maximum value, N_m say, at $\kappa = \kappa_m$. The value of κ_m depends in general on the numerical values of W, θ and ζ, estimates of which will be made in the next section. Meanwhile we note that by inspection $\kappa_m = 0$ in the simple case in which

$$(1+\theta)W + \tfrac{1}{2}\zeta U < 0, \tag{4.11}$$

which is always satisfied when acceleration-reaction effects are negligible (since $W = \phi\, dU/d\phi < 0$) and which we shall see in §5 is effectively always satisfied when the particles are solid. The disturbances of greatest relevance to stability questions are thus those of long wavelength; and for these the assumption of small gradients made in §3 is certainly valid. The condition that there should be some (small) values of κ for which $c_i > 0$ is

$$N_m = \frac{\{(1+\theta)W + \frac{1}{2}\zeta U\}^2}{(1+\theta)Q + \frac{1}{4}\zeta^2 U^2} > 1 \tag{4.12}$$

when (4.11) is satisfied.

It will be recalled that $W = \phi\, dU/d\phi$ is the velocity of propagation, relative to the particles, of the kinematic concentration waves that exist when $U^2/\gamma|\tilde{g}|a \ll 1$, and that Q, in the denominator of the fraction in (4.12), is the effective bulk modulus of elasticity of the particle configuration divided by the particle mass per unit volume of the mixture. Thus W and $Q^{\frac{1}{2}}$ are the speeds of propagation, relative to the particles, of two different kinds of non-dispersive wave of small amplitude which may exist separately under certain conditions. In the absence of acceleration-reaction effects (as in the case of a gas-fluidized bed), when N_m reduces to W^2/Q, the condition for growth of the amplitude of a disturbance at some wavenumbers is thus that the kinematic wave speed should exceed the dynamic wave speed. This strikingly simple criterion for instability is similar to that derived by Wallis (1962, 1969 §§6.5, 8.8) from general one-dimensional dynamical equations for a continuous medium that possesses a bulk elasticity modulus and that is acted on by a body force dependent on both the local velocity and the local density (in other words, a medium for which the momentum equation for a small disturbance to a uniform state contains on the right-hand side, like (3.14), terms proportional to $\partial\phi'/\partial x$, V' and ϕ'). Note however that in two-phase flow with acceleration-reaction effects Wallis's criterion for instability needs modification and that the criterion is of value only insofar as we know what physical processes are represented by Q.

When $N_m > 1$ the range of wavenumbers for which $c_i > 0$ is $0 < \kappa < \kappa_n$, where κ_n is the neutrally stable wavenumber for which N (as given by (4.10)) equals unity. After some manipulation we find

$$\kappa_n^2 = \frac{\gamma\tilde{g}}{\eta U}(N_m^{\frac{1}{2}} - 1)\frac{(1+\theta)W + \frac{1}{2}\zeta U}{(1+\theta)W - \frac{1}{2}\zeta U(N_m^{\frac{1}{2}} - 1)}, \tag{4.13}$$

where N_m is given by (4.12). It is probable that the wavenumber at which the growth rate κc_i has its maximum (when $N_m > 1$) differs from κ_n by a factor of order unity only, at any rate for values of N_m not far above unity.

We may investigate the values of c_r and c_i in the neighbourhood of the critical condition $N = 1$ by writing the inner square-rooted quantity in (4.7) and (4.8) as

$$(\tfrac{1}{4}\hat{P}^2 + \hat{Q})^2 + \left(\frac{\gamma\tilde{g}\hat{W}}{\kappa U}\right)^2\left(\frac{N-1}{N}\right)$$

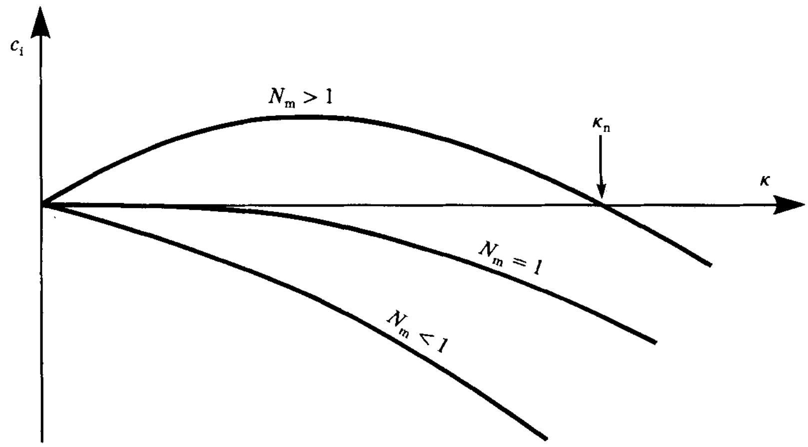

FIGURE 1. Sketch showing the nature of the dependence of c_i on κ near the origin when $|N-1| \ll 1$. N_m is the maximum value of N with respect to κ and occurs at $\kappa = \kappa_m = 0$.

and expanding in powers of the small quantity $N-1$. The relations (4.7) and (4.8) (with the choice of root corresponding to the positive sign in (4.6)) reduce to

$$c_r \approx \hat{U} - \hat{Q}^{\frac{1}{2}} - \frac{\frac{1}{2}\hat{P}^2\hat{Q}^{\frac{1}{2}}(N-1)}{\hat{P}^2+4\hat{Q}}, \quad c_i \approx \frac{\hat{P}\hat{Q}(N-1)}{\hat{P}^2+4\hat{Q}} \tag{4.14}$$

when $|N-1| \ll 1$. From (4.14) we see that the propagation velocity of the neutral disturbance, relative to the particles, is

$$-\hat{Q}^{\frac{1}{2}} - \frac{\frac{1}{2}\zeta}{1+\theta}U, \tag{4.15}$$

which is the common velocity of a kinematic concentration wave and the upwardly propagating dynamic wave when $\theta = \zeta = 0$. We also obtain the growth rate κc_i for nearly neutral disturbances as a function of κ and ϕ and the parameters specifying the properties of the particles. The general nature of the dependence of c_i on κ is evident from (4.14) and the separate dependences of $\hat{P}$ and N on κ given by (4.5) and (4.10) respectively, and is sketched in figure 1. Near the origin the curves in figure 1 have the linear form

$$c_i \sim \kappa \frac{U(1+\theta)}{\gamma\tilde{g}} \hat{Q}(N_m - 1). \tag{4.16}$$

5. The conditions for instability of a fluidized bed in numerical form

The foregoing analysis of the behaviour of plane-wave disturbances has revealed the physical processes promoting or opposing instability of the bed and has determined the conditions under which disturbances grow. There arises now the question whether the conditions found to be necessary for instability are realized in practice. This requires the conditions for instability to be put into numerical form. The conditions have been expressed in §4 in terms of various parameters of a homogeneous fluidized bed and we therefore proceed to make the best possible estimates of the numerical values of these parameters using either theoretical reasoning or observational data. Some of the estimates are rather crude and provisional because both theory and observational data are lacking at the present time.

We suppose now that the particles are solid spheres, of radius a. For particles of this kind there is evidence of the instability of both gas- and liquid-fluidized beds.

The most important of the parameters to be estimated is the mean particle velocity U in a homogeneous fluidized bed. For particles that exert only hydrodynamic forces on each other and are large enough for effects of Brownian motion to be negligible ($a > 1\ \mu\text{m}$), U/U_0 is a function of ϕ and of the flow Reynolds number, where U_0 is the velocity of an isolated particle in fluid at rest at infinity. Many measurements of this function have been made, and a widely used correlation of the data for solid particles is that associated with the names Richardson & Zaki, viz.

$$U(\phi) = U_0(1-\phi)^p, \tag{5.1}$$

where the power p varies monotonically with the Reynolds number $\mathbb{R}_0\,(= 2a|U_0|\rho_f/\mu)$ of the flow about an isolated particle, from about 5 at very small Reynolds number to about 2.5 at very large Reynolds number. A set of careful measurements of the mean fall velocity of sedimenting spheres in the small-Reynolds-number range which defines $U(\phi)/U_0$ empirically over the whole range of values of ϕ was made by Buscall *et al.* (1982) and is reproduced in figure 2. These measurements are represented well by the algebraic form (5.1) with $p = 5.5$, and we shall adopt this value for the case of small Reynolds numbers. The velocity U_0 depends on the particle radius and density, in a way which is determined by the Reynolds number. At small particle Reynolds number the Stokes-drag relation

$$U_0 = \frac{m\tilde{g}}{6\pi\mu a} = \frac{2a^2\rho_p\tilde{g}}{9\mu} \tag{5.2}$$

may be used, where μ is the fluid viscosity; otherwise an empirical relation between U_0 and a and ρ_p is needed.

If $U(\phi)$ has the form (5.1) then for $W(\phi)$ we have

$$W(\phi) = \phi\frac{\mathrm{d}U}{\mathrm{d}\phi} = -\frac{p\phi}{1-\phi}U = -pU_0\phi\,(1-\phi)^{p-1}, \tag{5.3}$$

which is negative everywhere and has a maximum magnitude at $\phi = p^{-1}$.

Another function of ϕ which occurs in the stability parameter N is HU^2, representing the mean-square particle velocity fluctuation in a homogeneous bed. This is almost completely unknown. One might make the simple assumption that the dimensionless function $H(\phi)$ varies quadratically between zero at $\phi \to 0$, when a particle is effectively isolated, and zero again at close packing ($\phi = \phi_{cp}$), when a particle is 'locked' into a falling cage of particles. The maximum value of H might be as large as 0.25, at which value the r.m.s. velocity fluctuation is 50% of the mean. On this rough tentative basis

$$H(\phi) \approx \frac{\phi}{\phi_{cp}}\left(1-\frac{\phi}{\phi_{cp}}\right). \tag{5.4}$$

The corresponding form of the function K is

$$K(\phi) = \frac{\mathrm{d}(\phi HU^2)}{\mathrm{d}\phi} = U_0^2(1-\phi)^{2p}H\left(2-\frac{2p\phi}{1-\phi}-\frac{\phi}{\phi_{cp}-\phi}\right). \tag{5.5}$$

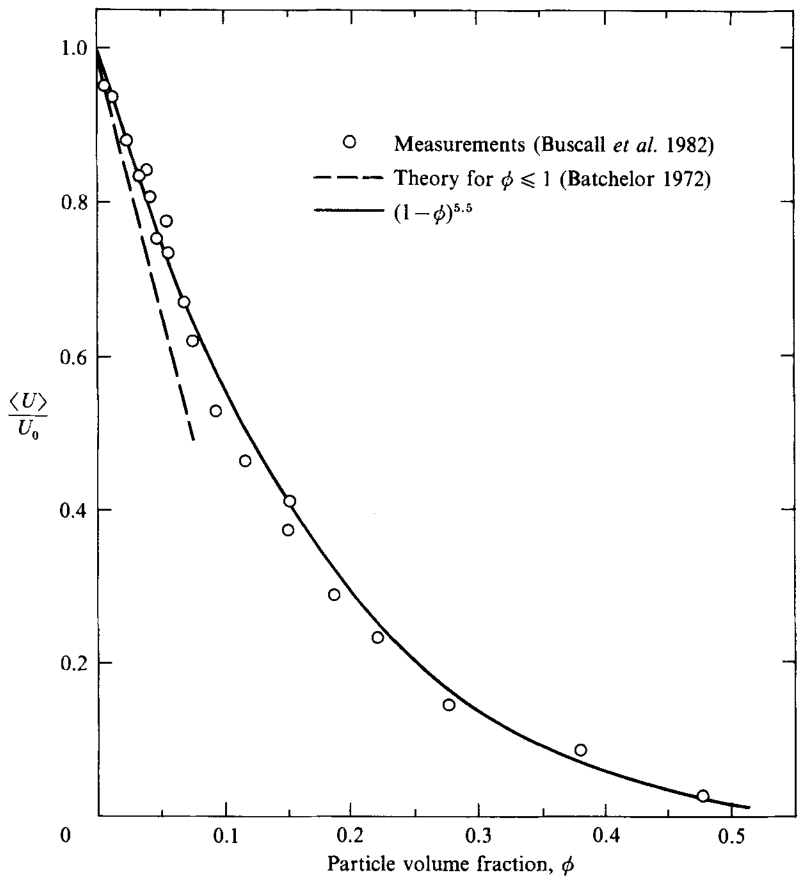

FIGURE 2. Measurements of the mean vertical velocity of polystyrene spheres of diameter 3.10 μm in water, made by Buscall *et al.* (1982). The continuous curve is the function $(1-\phi)^{5.5}$, and the broken line is a theoretical result (Batchelor 1972) for $\phi \ll 1$ and uniform pair probability density.

Two significant features of this expression, for the small-Reynolds-number value of p, are first that it is positive (corresponding to a positive contribution to the bulk modulus of elasticity of the particles) only when ϕ is less than about 0.14, and second that the magnitude of K/U_0^2 is everywhere small and less than 0.025 for $\phi > 0.1$. These features are mainly a consequence of the dominance of the factor U^2 in the expression for $\langle v^2 \rangle_h$.

The other contribution to the function Q defined in (3.11) (that is, to $(\phi \rho_p)^{-1}$ times the bulk modulus of elasticity) is $\gamma \tilde{g} D/U$, where D is the hydrodynamic gradient diffusivity of the particles. The concept of hydrodynamic diffusion of particles in a sedimenting dispersion is still novel, and there is a dearth of both theoretical calculations and observations of the value of D. It seems likely that the relevant velocity and length scales of the random motion of a particle are $|U|$ and a respectively, except perhaps near $\phi = 0$ and $\phi = \phi_{cp}$, in which event we may write D as the product of $a|U|$ and a dimensionless function of ϕ and the particle Reynolds number and the density ratio. There is little we can say about this function on theoretical grounds. We shall assume simply that

$$D = \alpha a |U| = \alpha a |U_0| (1-\phi)^p, \tag{5.6}$$

where α is a number of order unity, and suppose that α is constant for lack of information to the contrary. With regard to the numerical magnitude of α, a very recent experimental investigation of the diffusive spreading of the top of a cloud of sedimenting particles (at small Reynolds number) by Davis & Hassan (1988) suggests values of $D/a\bar{U}$ decreasing from about 12 to 8 as the volume fraction below the interface increases from 0.05 to 0.15, where $\bar{U}$ is the median value of U in the interface. The experiment gives a kind of weighted average diffusivity for all the values of ϕ present in the spreading interface and so does not specify a function $D(\phi)$, but is otherwise helpful in indicating the general magnitude of α for small values of ϕ. Unfortunately the values of α needed for the determination of the onset of instability of a fluidized bed that is being expanded are those for higher concentrations.

The diffusivity D appears in the expressions for Q and N_{m} in combination with $\gamma\tilde{g}/U$, and in place of (3.11) we may write

$$Q = \frac{\mathrm{d}(\phi H U^2)}{\mathrm{d}\phi} + \alpha\gamma a|\tilde{g}| \tag{5.7}$$

since the sign of U is the same as that of $\tilde{g}$ (both being positive in the case of particles that move downward relative to the fluid).

Another coefficient with the dimensions of diffusivity occurring in the stability parameter N is η, $\phi\rho_{\mathrm{p}}\eta$ being the particle viscosity. No theory or experimental data about η are available, and the best we can do is to conjecture that a and $|U|$ are the relevant length and velocity scales and that

$$\eta = \beta a|U|, \tag{5.8}$$

where β is a number of order unity which may vary with ϕ. The particle viscosity causes damping of disturbances with large wavenumber and so determines the maximum wavenumber at which growth occurs at a given value of N_{m} and also the wavenumber for which the growth rate is a maximum, but it plays no part in the condition for growth to occur at some wavenumbers and so is less important than D.

Finally there are the two acceleration-reaction coefficients θ and ζ, both of which are determined by the function $C(\phi)$ as indicated in (2.11). It is not known whether the expression (2.8) represents the acceleration reaction correctly, although we hope that it gives fluid-inertia effects of the right order of magnitude. The magnitude of the function $C(\phi)$ is expected to be of order unity at all values of ϕ, and that is the most we are entitled to assume, but since it is clearer to work with definite functions than with one specified only in order of magnitude a definite choice of $C(\phi)$ will be made. We shall choose an expression for $C(\phi)$ which Zuber (1964) suggested would represent approximately the virtual mass of a particle in a homogeneous dispersion in which the fluid motion is irrotational (which, as noted in §2, is unlikely to be a realistic assumption for a dispersion of solid particles). Zuber postulated that each particle may be supposed to move with velocity V in fluid enclosed by a stationary boundary representing the hydrodynamic effect of all the other particles. He took the outer boundary to be a sphere of radius b instantaneously concentric with the moving spherical particle, and an easy calculation of the kinetic energy of the fluid in irrotational flow then shows that

$$C(\phi) = \frac{1+2\phi}{2(1-\phi)} \tag{5.9}$$

if b is chosen as $a/\phi^{\frac{1}{3}}$ to give the right ratio of solid to fluid volume.

We shall use (5.9) as a basis for judging the relative importance of fluid-inertia effects on the stability parameters. With this expression for C we have from (2.11)

$$\theta = \frac{1+2\phi}{2(1-\phi)}\frac{\rho_f}{\rho_p}, \quad \zeta = \frac{3\phi}{4(1-\phi)^2}\frac{\rho_f}{\rho_p}. \tag{5.10}$$

Inspection of the expression for N in (4.10) shows that we need to know first whether $1+\theta$ differs significantly from 1. The acceleration reaction is always negligible when the fluidizing fluid is a gas, but for a liquid ρ_f/ρ_p typically lies between 0.1 and 0.5 and since according to (5.10) θ rises monotonically to $2\rho_f/\rho_p$ at the large particle concentration $\phi = 0.5$, it is clear that θ is not always small compared with unity. ζ is also a monotonically increasing function of ϕ, and is equal to $1.5\rho_f/\rho_p$ at $\phi = 0.5$. In §4 we saw that the wavenumber κ_m at which N has its maximum value is zero when the condition (4.11) is satisfied, and since according to (5.3) and (5.10)

$$\frac{\frac{1}{2}\zeta U}{(1+\theta)|W|} = \frac{3}{8p(1+\theta)(1-\phi)}\frac{\rho_f}{\rho_p}, \tag{5.11}$$

it appears that (4.11) is always satisfied when $\rho_p > \rho_f$. Henceforth we shall accept $\kappa_m = 0$ without qualification, with the consequence that the condition for growth at some (small) values of κ is $N_m > 1$ where N_m is given by (4.12).

With the help of (5.3), (5.4), (5.7) and (5.10) we may now write the expression (4.12) for N_m as

$$N_m = \frac{\left\{-\frac{p\phi U}{1-\phi}(1+\theta) + \frac{3\phi U}{8(1-\phi)^2}\frac{\rho_f}{\rho_p}\right\}^2}{\left\{\frac{d(\phi H U^2)}{d\phi} + \alpha\gamma a|\tilde{g}|\right\}(1+\theta) + \left\{\frac{3\phi U}{8(1-\phi)^2}\frac{\rho_f}{\rho_p}\right\}^2}, \tag{5.12}$$

which is not as opaque as it might seem since the dependence on ϕ of each term in the numerator and denominator is fairly simple. N_m can be regarded as a function of three dimensionless variables, the particle concentration ϕ representing the operating conditions of the fluidized bed, the Froude number

$$\xi = \frac{U_0^2}{\gamma a|\tilde{g}|} \tag{5.13}$$

representing the particle properties, and the density ratio ρ_p/ρ_f which determines the magnitude of fluid-inertia effects (and also the numerical value of U_0). As we have expected from the discussion in §3, ξ is the key quantity determining whether a uniform fluidized bed is unstable under some operating conditions.

Further consideration of the conditions for instability in numerical form is best given to the two cases of gas- and liquid-fluidized beds separately.

The case of a gas-fluidized bed

The parameters θ and ζ are negligibly small in this case, and the expression (5.12) reduces to

$$N_m = \frac{\frac{p^2\phi^2}{(1-\phi)^2}\left(\frac{U}{U_0}\right)^2}{\frac{1}{U_0^2}\frac{d(\phi H U^2)}{d\phi} + \frac{\alpha}{\xi}}. \tag{5.14}$$

Trial calculations show quickly that values of N_m near unity correspond to values of ξ and of a such that the Reynolds number of the flow about a particle is small. Let us therefore assume small-Reynolds-number flow provisionally. In that event the power p in the empirical expression (5.1) for $U(\phi)$ can be chosen as 5.5 on the basis of the observations shown in figure 2. Also, we can put $\gamma = 1$. The numerator in (5.14) then has a maximum value of 0.164 at $\phi = \phi_m = p^{-1} = 0.182$. Hence values of the denominator of (5.14) near 0.16 are needed for N_m to be near the critical value unity. Now according to the estimate of the function H given in (5.4) and used in (5.5) (an estimate that needs to be checked by observation of the mean-square fluctuation in particle velocity), the magnitude of the first term in the denominator of (5.14) is smaller than 0.025 at all values of ϕ above 0.1. It appears that the second term in the denominator of (5.14) is appreciably larger than the first at conditions near critical. The condition for instability of a gas-fluidized bed may thus be written to a fair approximation as

$$N_m \approx \frac{W^2}{\alpha g a} = 30.25\phi^2(1-\phi)^9 \frac{\xi}{\alpha} > 1. \tag{5.15}$$

It is worthwhile to note in passing the implication of this tentative conclusion, viz. that the contribution to the function Q in (5.7) (or to the bulk modulus of elasticity of the particles) due to the effective force exerted hydrodynamically between particles on the two sides of a horizontal plane surface is positive and larger in magnitude than that due to transport of particle momentum across the surface. The effective repulsive force between particles, which is related to the particle diffusivity, evidently provides the primary opposition to growth of a disturbance to the particle concentration in the bed. The particle diffusivity, which does not appear in any of the governing equations used in previous analyses of the stability of a fluidized bed, is seen to be a key quantity.

As noted previously, we are regarding the diffusivity coefficient α as not varying appreciably with ϕ, for lack of information to the contrary, in which case the expression for N_m in (5.15) has a maximum with respect to ϕ at $\phi = \phi_m = 0.182$ and

$$(N_m)_{\phi=\phi_m} = \frac{\xi}{6.09\alpha}, \quad = \frac{\xi}{\xi_c} \quad \text{say.} \tag{5.16}$$

The condition for growth of disturbances at some values of ϕ is thus that ξ exceeds a critical value ξ_c, where $\xi_c = 6.09\alpha$. A criterion such that U_0^2/ag should exceed a number of order unity† was to have been expected on dimensional grounds, since U_0, a and g are the primary relevant parameters of the problem and U_0^2 represents the particle inertia forces which must exceed a certain magnitude for instability.

Values of $N_m^{\frac{1}{2}}$ as a function of ϕ (according to (5.15)) are shown in figure 3 for various values of ξ/α, and we may imagine each one of these curves as giving the values of N_m at the different stages of expansion of a fluidized bed. When the fluid speed is just large enough to fluidize the bed, ϕ is close to its maximum value and $N_m < 1$ for all the curves shown in figure 3, As the fluid speed is increased the bed expands and ϕ decreases. If $\xi < \xi_c$ the value of N_m increases to a maximum which is less than unity before decreasing to zero, and the bed is stable for all speeds of the fluidizing fluid. On the other hand, if

$$\xi\left(= \frac{U_0^2}{ag}\right) > \xi_c\,(= 6.09\alpha), \tag{5.17}$$

the value of N_m reaches the critical value unity at a certain value of ϕ, and then

† A criterion for bubbling of a fluidized bed of this form has been put forward on empirical grounds by several authors, beginning with Wilhelm & Kwauk (1948).

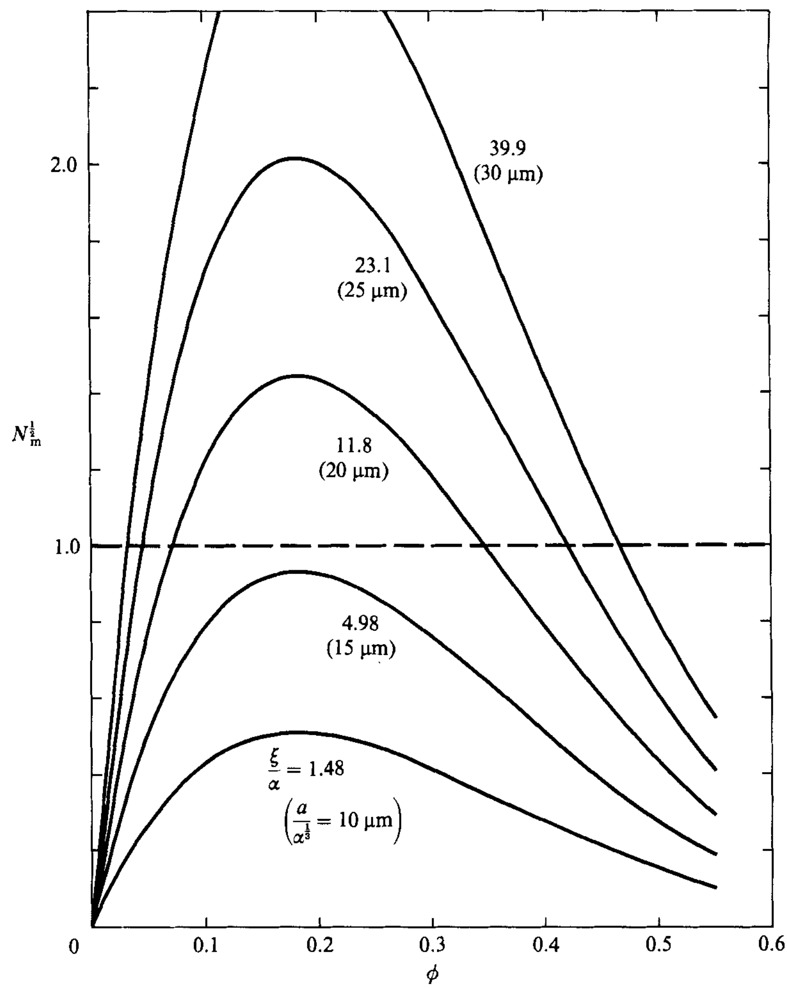

FIGURE 3. The stability parameter N_m as a function of ϕ for various values of the particle Froude number ξ in the case of a gas-fluidized bed (for which the particle Reynolds number is small). If the fluid is air at normal temperature and pressure and $\rho_p = 1\ \text{gm/cm}^3$, the values of ξ correspond to values of the particle radius a shown in parenthesis on the curves. The diffusivity coefficient α is expected to be of order unity.

exceeds it, showing that the bed becomes unstable when it is been expanded sufficiently. The bed becomes stable again at even higher fluidizing speeds and small values of ϕ, and it is evident from the ϕ-dependence of the numerator of the general expression for N_m in (5.12) that this holds for fluidized beds at any particle Reynolds number.†

The relation between ξ and the particle radius a may now be seen from the Stokes-drag law (5.2) to be

$$\xi = \frac{U_0^2}{ag} = \frac{4a^3\rho_p^2 g}{81\mu^2}. \tag{5.18}$$

Combination of (5.17) and (5.18) then shows that the following relation between the

† Foscolo & Gibilaro (1984) say that 'bubbling beds are known to revert to homogeneous behaviour at high voidages'.

properties of the particles and the fluid must be satisfied by a fluidized bed that becomes marginally unstable at $\phi = \phi_m$:

$$\frac{a^3\rho_p^2 g}{\mu^2} = 123\alpha. \tag{5.19}$$

The value of α is needed for the practical interpretation of figure 3 and the relation (5.19). We know too little about the particle diffusivity at the values of ϕ that are relevant to the onset of instability to be able to chose a specific value of α, and it seems preferable at this stage to leave it as an unspecified factor as shown on the curves in figure 3. α is believed to be of order unity in the algebraic sense.

If we assume the value of the viscosity μ to be as for air at normal temperature and pressure, and take $\rho_p = 1\ \mathrm{gm/cm^3}$ as a representative particle density and $g = 981\ \mathrm{cm/s^2}$, (5.18) becomes

$$\xi = 1.48 \times 10^6\,(a\,\mathrm{mm})^3 \tag{5.20}$$

from which the values of a shown on the curves in figure 3 have been found. The largest particle radius for which the bed is not unstable at some value of ϕ is found from (5.19), with these same particular values of ρ_p and μ, to be

$$a = 16.0\,\alpha^{\frac{1}{3}}\,\mu\mathrm{m}. \tag{5.21}$$

If $\alpha = 1$ this critical size is 16.0 μm, and if $\alpha = 5$ it is 27.4 μm. For any finer particles the fluidized bed (and a cloud of sedimenting particles) is stable at all concentrations. For particles of these radii the assumption of small flow Reynolds number is permissible.

The range of wavenumbers for which growth occurs when $N_m > 1$ is $0 < \kappa < \kappa_n$, and we see from (4.13) and (5.8) that for a gas-fluidized bed κ_n is given by

$$(\kappa_n a)^2 = \frac{ag}{\beta U^2}(N_m^{\frac{1}{2}} - 1) = \frac{N_m^{\frac{1}{2}} - 1}{\beta\xi(1-\phi)^{11}}. \tag{5.22}$$

At the value of ϕ at which N_m is a maximum with respect to ϕ, this becomes

$$(\kappa_n a)_{\phi=\phi_m} = \frac{1.22\{(\xi/\xi_c)^{\frac{1}{2}} - 1\}^{\frac{1}{2}}}{(\alpha\beta)^{\frac{1}{2}}\,(\xi/\xi_c)^{\frac{1}{2}}}, \tag{5.23}$$

which is shown graphically by the continuous curve in figure 4. Incidentally this diagram makes more specific, for the case of a gas-fluidized bed free from the complications of acceleration-reaction and Reynolds number effects, the rather vague general remarks in §§3 and 4 about the roles of the two Froude numbers U^2/ga and $\kappa U^2/g$. As $\xi(= U_0^2/ga)$ becomes larger, the maximum wavenumber for which growth occurs decreases to zero as $\xi^{-\frac{1}{4}}$ (the continuous curve) and the maximum value of $\kappa U_0^2/g$ for which growth occurs increases as $\xi^{\frac{3}{4}}$ (the broken curve). If both U_0^2/ga and $\kappa U_0^2/g$ are small compared with unity the bed is stable, and if both are large, and of the same order of largeness, it is again stable. Similar remarks apply to U^2/ga and $\kappa U^2/g$ at a particular value of ϕ.

At conditions near the critical (that is, when $N - 1 \ll 1$), the velocity of propagation of a disturbance relative to the particles is seen from (4.15) to be $-Q^{\frac{1}{2}}$ in a gas-fluidized bed. With neglect of the first term in the expression (5.7) for Q and the use of (5.13) this becomes

$$c_r - U \approx -\left(\frac{\alpha}{\xi}\right)^{\frac{1}{2}} U_0. \tag{5.24}$$

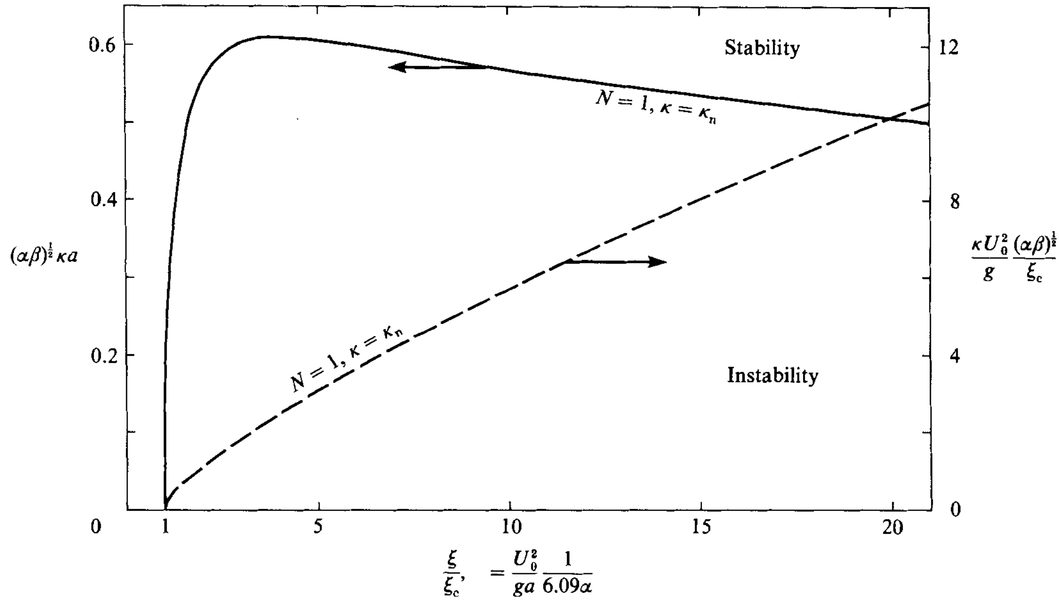

FIGURE 4. The regions of stability and of instability in the (κ, ξ)-plane for a gas-fluidized bed at $\phi = \phi_m = 0.182$. κ_n is the wavenumber at which the growth rate is zero. For points beneath the continuous curve (to which the left-hand ordinate scale is relevant) the bed is unstable; and similarly for the broken curve associated with the right-hand ordinate scale and a different non-dimensionalization of κ.

It is noteworthy that $c_r - U$ is independent of both κ and ϕ in this approximation. In the case of a bed that becomes marginally unstable at just one value of ϕ we may put $\xi = \xi_c = 6.09\alpha$, giving $c_r - U = -0.40U_0$.

Finally, there is the magnitude of the growth rate. The fractional increase in amplitude of a growing wave in the time taken by a wave crest to propagate a distance of one wavelength relative to the particles is

$$\frac{2\pi c_i}{|c_r - U|} = \frac{2\pi P Q^{\frac{1}{2}}(N-1)}{P^2 + 4Q} \tag{5.25}$$

if we use the approximations (4.14) valid when $|N-1| \ll 1$ and the approximations appropriate to a gas-fluidized bed. This function of κ is zero at $\kappa = 0$, increases linearly with κ initially, and then turns down to become zero again at the neutrally stable wavenumber κ_n, like the sketch of c_i in figure 1. In the linear range c_i is given by (4.16) (in which $\hat{Q}$ may be replaced by $\alpha a g$) and we obtain a crude estimate of the maximum value of (5.25) with respect to κ by evaluating (4.16) at $\kappa = \frac{1}{2}\kappa_n$, the result being

$$\pi(N_m - 1)(N_m^{\frac{1}{2}} - 1)^{\frac{1}{2}}(\alpha/\beta)^{\frac{1}{4}}. \tag{5.26}$$

This estimate for the maximum value of $2\pi c_i/|c_r - U|$ depends only on N_m, and is equal to 0.194 when $N_m = 1.2$ if we put $\alpha = \beta$, meaning that the amplitude of a disturbance which originates near the bottom of a fluidized bed for which $N_m = 1.2$ has increased by a factor $\exp(0.194 \times 10) = 6.96$ at a height of 10 wavelengths above the bottom.

The case of a liquid-fluidized bed

The conditions for instability are more complicated here, for two reasons, the first of which is that the relation between U and ϕ and a depends on the Reynolds number of the flow about a particle. The presence of the fluid viscosity μ in the small-$\mathbb{R}$ expression (5.18) for the particle Froude number ξ shows that the critical value of ξ occurs at a much larger value of the particle radius a in a water-fluidized bed than in an air-fluidized bed, more than 10 times larger according to this expression, and a trial calculation makes it clear that at conditions near the critical for instability the Reynolds number is no longer small. Nor could we have expected to be able to ignore the inertia of the fluid in a situation in which the inertia of the particles, with density not greatly different from that of the fluid, is important. Consequently it is necessary to return to (5.1) and choose for the power p a smaller and Reynolds-number-dependent value and to replace the Stokes-drag relation (5.2) by an empirical Reynolds-number-dependent drag coefficient.

New numerical estimates which allow for Reynolds-number effects are presented in table 1 in a form that enables application to be made to a fluidizing fluid of any density and viscosity and particles of any density and size. For the purpose of this table the independent variable is the Reynolds number $\mathbb{R}_0(=2a|U_0|\rho_f/\mu)$ of the flow about an isolated falling particle, values of which are shown in column 1. Column 2 of the table shows the corresponding value of the power p in the Richardson–Zaki correlation (5.1) for U as a function of ϕ as given in the text by Wallis (1969) (with some upward adjustment of the values of p at the smaller Reynolds numbers for conformity with recent sedimentation observations). There are many empirical relations and correlations of data for the drag coefficient of an isolated rigid sphere (defined as $C_{D0}=m|\tilde{g}|/\frac{1}{2}\pi a^2\rho_f U_0^2$) over various restricted ranges of Reynolds number (see Clift, Grace & Weber 1978, chap. 5), but none that are accurate over the whole range from small to moderately large values of $\mathbb{R}_0$. My purpose here is to illuminate the various physical processes involved in instability rather than to provide definitive numerical relationships, and for convenience I shall chose a single formula for use over the range $0<\mathbb{R}_0\leqslant 2000$, which is as high as we need go for present purposes. In a recent study of instability of a fluidized bed, Foscolo & Gibilaro (1984) recommended the 'correlation'

$$C_{D0}=(0.63+4.90\mathbb{R}_0^{-\frac{1}{2}})^2, \tag{5.27}$$

which they attribute to Dallavalle (1948), and I have used (5.27) to obtain the numbers in column 3. Column 4 then gives values of the drag-slope parameter γ defined by (3.3) (with the empirically supported assumption that γ does not vary much with ϕ and so can be evaluated at $\phi=0$), which is equal to $1+0.63C_{D0}^{-\frac{1}{2}}$ when the empirical expression (5.27) for C_{D0} is used. The particle radius is given in terms of C_{D0} and $\mathbb{R}_0$ by

$$a=\left(\frac{3\rho_f\nu^2C_{D0}\mathbb{R}_0^2}{32\rho_p|\tilde{g}|}\right)^{\frac{1}{3}}, \tag{5.28}$$

from which we obtain column 5, where $\nu=\mu/\rho_f$ is the kinematic viscosity of the fluid. The relation between the particle Froude number ξ and the drag coefficient C_{D0} is

$$\xi=\frac{U_0^2}{\gamma a|\tilde{g}|}=\frac{8\rho_p}{3\rho_f\gamma C_{D0}}, \tag{5.29}$$

1	2	3	4	5	6	7	8
$\mathbb{R}_0$	p	C_{D0}	γ	$a\left(\frac{\rho_p\lvert\tilde{g}\rvert}{\rho_f\nu^2}\right)^{\frac{1}{3}}$	$\xi\frac{\rho_f}{\rho_p}$	$\theta_m\frac{\rho_p}{\rho_f}$	$\frac{\xi_c(1+\theta_m)}{\alpha}$
$\ll 1$	5.5	$24\,\mathbb{R}_0^{-1}$	1.0	$1.31\,\mathbb{R}_0^{\frac{1}{3}}$	$\frac{1}{9}\,\mathbb{R}_0$	0.833	6.09
1.0	5.0	30.6	1.11	1.419	0.079	0.875	5.96
2	4.7	16.77	1.15	1.844	0.139	0.905	5.87
5	4.3	7.96	1.22	2.652	0.275	0.955	5.74
10	4.0	4.75	1.29	3.541	0.436	1.000	5.62
20	3.7	2.98	1.37	4.813	0.655	1.056	5.48
50	3.3	1.75	1.48	7.427	1.032	1.152	5.26
100	3.0	1.252	1.56	10.55	1.366	1.250	5.06
200	2.7	0.953	1.65	15.29	1.696	1.382	4.82
500	2.5	0.721	1.74	25.66	2.126	1.500	4.63
1000	2.4	0.616	1.80	38.65	2.401	1.571	4.52
2000	2.4	0.547	1.85	58.98	2.635	1.571	4.52

TABLE 1. Parameters of a uniform bed of solid spheres fluidized by either a gas (in which case only the first row of numbers is relevant) or a liquid. The condition that the bed is unstable for some values of κ and ϕ is $\xi > \xi_c$.

from which column 6 is obtained. Thus columns 5 and 6 specify a unique relationship between ξ and a when ρ_p/ρ_f and ν are known.

Explanation of the remaining columns 7 and 8 in table 1 raises the second complicating feature of a liquid-fluidized bed, which is that acceleration-reaction effects are not obviously negligible. We return to the general expression for N_m in (5.12), and note that estimates of the term $d(\phi HU^2)/d\phi$ from the suggested expression (5.5) show that it makes a relatively small contribution to the denominator of (5.12) at conditions near critical (that is, when the numerator and denominator are nearly equal) and as a rough appriximation, which is less accurate than in the case of a gas-fluidized bed, may be neglected. Moreover, since the acceleration-reaction terms in (5.12) are never dominant, the values of ξ (defined in (5.13)) at which N_m is near unity will be of the same general magnitude as in the case of a gas-fluidized bed. The numbers in column 6 of table 1 make it clear that this is possible for values of ρ_p/ρ_f mostly well above unity, which tells us that the acceleration-reaction terms may be small at conditions near the critical. We shall therefore drop provisionally the two terms in (5.12) coming from the expression (5.10) for ζ, and check later that the results are consistent with this. Thus (5.12) reduces to

$$N_m = \frac{\xi}{\alpha}(1+\theta)\,p^2\phi^2(1-\phi)^{2(p-1)}; \tag{5.30}$$

the other acceleration-reaction parameter, θ, is not so likely to be negligible and in any event its retention does not cause much difficulty.

As before, we are interested in the maximum value of N_m as ϕ varies. θ increases monotonically with ϕ according to (5.10), but the total variation is only about $1.5\rho_f/\rho_p$, which is almost always less than unity. Thus, if α is again assumed to be independent of ϕ, the location of the maximum of N_m is determined mainly by the opposition of the two factors ϕ^2 and $(1-\phi)^{2p-1)}$ and so is near $\phi = \phi_m = p^{-1}$. The value of the maximum is approximately

$$(N_m)_{\phi=\phi_m} = \frac{\xi}{\alpha}(1+\theta_m)\,(1-p^{-1})^{2(p-1)}, \tag{5.31}$$

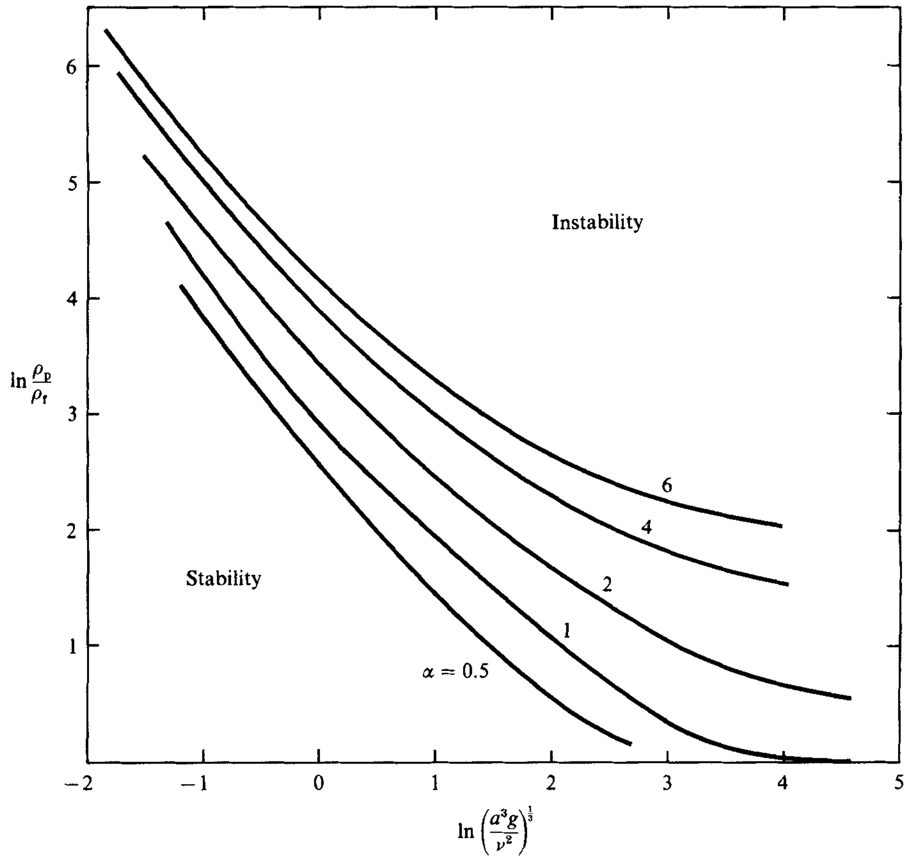

FIGURE 5. The relation between the particle density and radius for a bed that is marginally unstable at one value of ϕ, for five different assumed values of the diffusivity coefficient α. The unit of length for the radius, viz. $(\nu^2/g)^{\frac{1}{3}}$, is 0.0468 mm in the case of a bed fluidized with water and 0.284 mm in the case of air.

where
$$\theta_m = \frac{p+2}{2(p-1)}\frac{\rho_f}{\rho_p}, \tag{5.32}$$

and the condition for growth of disturbances at some values of ϕ near $\phi = \phi_m$ (and at some wavenumbers near $\kappa = 0$) is

$$\xi > \xi_c = \alpha \frac{(1-p^{-1})^{2(1-p)}}{1+\theta_m}. \tag{5.33}$$

Column 7 of table 1 shows the values of $\theta_m \rho_p/\rho_f$ according to (5.32), and then in column 8 there are values of $\xi_c(1+\phi_m)/\alpha$ according to (5.33).

The question whether a fluidized bed becomes unstable at some values of ϕ may now be answered by comparing columns 6 and 8 to see whether ξ is larger or smaller than the critical value ξ_c. This comparison requires a knowledge of the density ratio ρ_p/ρ_f, the particle radius a, and the kinematic viscosity of the fluid ν, from which the value of $a(\rho_p|\tilde{g}|/\rho_f \nu^2)^{\frac{1}{3}}$ may be calculated. This specifies the relevant row (or interpolation between two rows) in table 1, and the entries in columns 6, 7 and 8 are noted. The value of ξ follows from the entry in column 6, and then after a selection of the value of the diffusivity coefficient α, the evidence for which is scanty at the moment but may be expected to improve, ξ_c is found from the entries in columns 7 and 8. If ξ is very close to ξ_c, the fluidized bed is marginally unstable at a degree of expansion corresponding to $\phi = \phi_m$, and if $\xi > \xi_c$, the bed is unstable for a range of values of ϕ that includes $\phi = \phi_m$ and which increases in extent as ξ/ξ_c increases.

The way in which the two non-dimensional parameters ρ_p/ρ_f and $a(g/\nu^2)^{\frac{1}{3}}$ control the stability of the bed is shown by figure 5. Each of the curves in figure 5 is a line of neutral or marginal stability on which $\xi = \xi_c$ for a particular assumed value of α, and was obtained by first determining ρ_p/ρ_f from a comparison of columns 6 and 8 and then $a(g/\nu^2)^{\frac{1}{3}}$ from column 5. Continuation of the curves in figure 5 to larger values of $a(g/\nu^2)^{\frac{1}{3}}$ would require an expression for C_{D0} that is more accurate at the larger Reynolds numbers than (5.27).

Now we know that the value of ρ_p/ρ_f for a marginally unstable bed is seldom less than 2 we may return to (5.12) and confirm that the two terms coming from the parameter ζ are indeed relatively small (the one in the numerator of (5.12) being about 0.05 times the term retained when $\rho_p/\rho_f = 2$) as assumed provisionally. On the other hand it is evident from the numbers in column 7 of table 1 that neglect of the acceleration-reaction parameter θ would be justified quantitatively only as a rough approximation. The acceleration reaction has no qualitative influence on the stability properties of a liquid-fluidized bed of solid particles, but further theoretical investigation of the direct effect of fluid inertia on the force acting on particles in unsteady motion would be useful for quantitative purposes.

Estimates of the range of wavenumbers for which growth occurs when $N_m > 1$, the wave speed of disturbances, and the growth rate may be made for a liquid-fluidized bed with straightforward modifications of the procedure used for a gas-fluidized bed.

6. Recapitulation, and some connections with previous work

It has been found possible to formulate equations governing the unsteady one-dimensional mean motion of the particles in a gas- or liquid-fluidized bed without invoking any major untestable hypotheses. This is rare in the literature of two-phase flow and no doubt is a consequence of the assumed one-dimensionality of the mean motion. The form of these equations that is appropriate for a perturbed uniform fluidized bed is given in (2.2) and (3.10) and contains five parameters, viz. U, H, D, η and C, all expected to be functions of ϕ and the particle Reynolds number and the density ratio ρ_p/ρ_f. The mean particle velocity $U(\phi)$ is known empirically, and it should be possible to make measurements of $H(\phi)$ (the non-dimensional mean-square particle velocity fluctuation in a uniform fluidized bed) although there are few published data. D is the gradient diffusivity of the particles resulting from their hydrodynamically induced velocity fluctuations, and is a fundamental measurable quantity likely to be relevant to a variety of processes and phenomena associated with particle dispersions; recognition of its important role in the present problem is a key feature of this paper. Both H and D occur in the expression for the effective bulk modulus of elasticity of the particle configuration (see (3.11)), and it appeared from rough estimates of their magnitudes that the term containing H is relatively small at conditions near critical for instability. This is important for the later stability calculations, and confirmation of the estimates is desirable. The fourth parameter η, the particle viscosity, will not be easy to measure, but approximate calculations like those made in the kinetic theory of gases may yield a reasonable estimate; fortunately η plays only the secondary role of damping the disturbances with large wavenumbers. The logical foundation of the terms in the equation of motion containing the acceleration-reaction parameter C is not satisfactory, but fortunately it seems that ρ_p/ρ_f is usually well above unity for marginally unstable fluidized beds and that acceleration-reaction effects have only a minor quantitative influence on the conditions for instability of a liquid-fluidized bed of solid particles. Thus U and D are the most relevant parameters in the stability problem, and data are lacking only for D.

The first calculations of the behaviour of a small disturbance to a uniform fluidized bed were made independently by Ruckenstein (1962), Jackson (1963) and Pigford & Baron (1965). In the introduction to his paper Jackson wrote: 'It is the object of the present work to develop a set of equations of motion for a fluidized bed based on a realistic picture of the system as an assembly of particles in interaction with a flowing fluid', which is precisely the approach of the present paper. Jackson began with a momentum balance equation like (2.3) but omitted the first and fourth terms on the right-hand side representing the transfer of momentum across the surface of the control volume and the interparticle force exerted across that surface. This corresponds to putting H, η and D equal to zero in (3.10), with the consequences that $Q = 0$, N is infinite, and the growth rate κc_i always has one positive root with a maximum at $\kappa \rightarrow \infty$. Despite their very simple form, Jackson's equations for the behaviour of a small disturbance revealed the possibility of growth due to the effect of particle inertia and some comparative features such as a larger growth rate for a gas- than for a liquid-fluidized bed, in accordance with observation. The present paper is in direct succession to this pioneering work by Jackson and makes it more complete.

Virtually all later investigations of the behaviour of a small disturbance to a uniform fluidized bed (see for example Murray 1965; Molerus 1967; Anderson & Jackson 1968; Garg & Pritchett 1975; Homsy, El-Kaissy & Didwania 1980; Needham & Merkin 1983; and, for a review, Jackson 1985), as well as the early paper by Pigford & Baron (1965), have taken a rather different line and have used governing equations that are to some extent hypothesized rather than 'based on a realistic picture of the system as an assembly of particles in interaction with a flowing fluid'. The usual procedure has been to write the equation of motion for each of the two 'phases' of the system, the particles and the fluid, as if they were continuous media occupying the whole space, with inclusion of a body force representing the effect of the presence of the other phase, and to ascribe physical and rheological properties to each of the two media on a heuristic basis. Some of these rheological properties of the two hypothetical continuous media cannot be deduced or measured, because they are not well-defined physically. One can go part way towards finding an equation of motion for one of the phases which formally resembles an equation for a continuum by taking an average over the volume occupied instantaneously by that phase in the manner described by Drew (1983), but his procedure achieves rigour at the cost of introducing the intractable problem of closure of averages of a complicated kind.

It is not possible to describe any of the quantitative results obtained in these papers in terms of the present analysis, because the governing dynamical equations are different and contain different parameters. Many of the previously assumed governing equations contain no term representing resistance to compression of the particle configuration, and so yield the result that a fluidized bed is always unstable.

However Garg & Pritchett (1975) recognized the close connection between the degree of instability of the bed and the assumed magnitude of the effective bulk modulus of elasticity of the particles. Since this work there has been speculation about possible physical origins of a large effective bulk modulus of elasticity of the fine powder particles which are known to be stable for at least small degrees of expansion of a gas-fluidized bed.

A quite different stream of work has started from Wallis's (1962, 1969) perceptive deduction from simple model equations that the criterion for instability is that the kinematic wave speed exceeds the dynamic wave speed (relative to the particles in both cases), a criterion that we have shown to be correct for the more complete and more specific form of the governing equations for a fluidized bed if acceleration-reaction effects are negligible. Quantitative use of this criterion requires some insight into the physical origin of the effective elasticity of the particle configuration and a calculation (or an appeal to observational data) of the numerical value of the bulk modulus as a function of ϕ. Attempts to calculate the effective bulk modulus of elasticity of the particle configuration from heuristic mechanical models have been made (Verloop & Heertjes 1970; Foscolo & Gibilaro 1984; Foscolo & Gibilaro 1987), but all these calculations are flawed by the misconception that the elasticity of the particle configuration is related to the dependence of the mean fluid drag force on the particle concentration. What is needed is an estimate of the dependence of the particle stress, i.e. the rate of transfer of particle momentum plus the force exerted between the particles on the two sides of unit area of a horizontal surface, on the concentration. A calculation of the particle stress would be very difficult, and it is fortunate that, as seen herein, the more important of the two contributions to this stress is related to the measurable gradient diffusivity of particles.

Wallis initially likened the crisis resulting from equality of the kinematic and dynamic wave speeds to the formation of a shock wave when the speed of a body through a gas reaches sonic speed, and the notion that bubbles are in some way an outcome of the formation of shocks or discontinuities in particle concentration has been taken up by some authors (Verloop & Heertjes 1970; Fanucci, Ness & Yen 1979). It is undoubtedly true that nonlinear terms in the particle-conservation equation lead to steepening of the rearward-facing slopes in the concentration distribution and to the gradual formation of a discontinuity, as Kynch (1952) showed, but it does not seem likely that growth of small disturbances in a uniform fluidized bed, which is a linear process, is connected with the formation of discontinuities. The existence of dynamic waves is not actually necessary for instability of a fluidized bed, as Jackson (1963) unintentionally showed from equations containing no term equivalent to a pressure gradient in a gas. Growth of small disturbances in a bed requires only the existence of kinematic waves and particle inertia, and results from a change in the phase relationship between fluctuations in the mean particle velocity and the concentration brought about by inertia forces. Elasticity of the particles (that is, resistance to confinement to a smaller volume of space) hinders the growth by dispersing concentration fluctuations, and, if the bulk modulus is large enough, may suppress it. Analogies such as sound waves in a gas and traffic flow (Wallis 1969) seem to be rather wide of the mark.

A quantitative comparison of the results of the present theory with observation is outside the scope of this paper, but we may note that the specific criterion for instability found by Foscolo & Gibilaro (1984) and compared with numerous observations happens to be not very different numerically from the criterion

$N_m > 1$ found herein. Their version of Wallis's criterion for growth of disturbances, in my notation, is

$$0.16p^2\phi(1-\phi)^{2(p-1)}\left(\frac{U_0^2}{a\tilde{g}}\right) > 1, \tag{6.1}$$

and the ratio of the left-hand side of this inequality to the expression for N_m given in (5.30) is

$$\frac{0.16\gamma\alpha}{\phi(1+\theta)}. \tag{6.2}$$

(The near identity of the algebraic forms of the two criteria is not too surprising, since Foscolo & Gibilaro have the correct expression for the kinematic wave speed, and the dynamic wave speed is likely to be found to be proportional to $(a\tilde{g})^{\frac{1}{2}}$ on dimensional grounds.) Since γ and $1+\theta$ lie between 1 and 2 and α is of order unity, the ratio (6.2) is not very far from unity for values of ϕ between 0.1 and 0.5. Foscolo & Gibilaro made a comparison of their criterion with all the available observations of the occurrence or non-occurrence of bubbling (assumed to occur whenever a bed is unstable†) in both gas- and liquid-fluidized beds, and found that 'the agreement is quite satisfactory'. Reference should be made to Foscolo & Gibilaro (1984) for the details. The data do not define the onset of bubbling sharply, nor is the value of α in my theory known to better than the order of magnitude, so the agreement found by Foscolo & Gibilaro implies only that my instability criterion is generally compatible with observations of the occurrence of bubbling.

The predictions of the present theory concerning the properties of amplified disturbances (none of which are given by Wallis's criterion), such as the wavelength for maximum growth rate, the growth rate, and the disturbance wave speed, may also be of some interest in practice, and can be calculated without difficulty when the values of α and, less importantly, of η are known or can be estimated. Some valuable direct observations of these properties of growing disturbances in several different liquid-fluidized beds were made by El-Kaissy & Homsy (1976). However, a comparison with the present theory is not yet possible, because the values of ϕ in their experiments were larger than 0.52 in every case and the values of α at these very high concentrations are quite unknown (and may not be approximately constant as provisionally assumed herein).

7. Index of symbols

I have not followed the rather elaborate notation relating to fluidized beds that is commonly used by chemical engineers, and consequently the paper may at first be a little difficult to read. The following index showing the meaning of a symbol or the number of the equation in which it is first introduced and defined may help.

Some of the symbols must be interpreted in the light of the conventions regarding the reference frame, viz.

(i) the vertical position coordinate x and the vertical velocity component are positive when directed downwards, like gravity;

(ii) the axes of reference are such that the mean flux of material volume across a horizontal plane surface is zero, so the mean particle velocity in a uniform fluidized bed is a downward velocity equal in magnitude to what is usually called the

† This assumption may not always be valid. El-Kaissy & Homsy (1976) have observed plane-wave disturbances with increasing amplitude in liquid-fluidized beds which do not evolve into bubble-like structures.

'superficial fluid velocity' in a real fluidized bed referred to axes fixed in the distributor plate.

Roman symbols

a	particle radius
B	(2.9)
c	(4.1)
C	(2.8)
C_D	(3.1)
D	(2.9)
F_h	(3.1)
$\tilde{g}$	$= g(\rho_p - \rho_f)/\rho_p$
H	(3.5)
K	(3.15)
m	particle mass
n	particle number density
N	(4.10)
N_m	value of N at $\kappa = \kappa_m$
p	(5.1)
$\hat{P}, \hat{Q}$	(4.5)
Q	(3.11), (5.7)
$\mathbb{R}$	(3.2)
S	(2.12)
V	mean particle velocity
U	--- in a uniform bed
v	velocity fluctuation
V'	(3.12)
W	(3.15)
$\hat{U}, \hat{W}$	(4.5)

Greek symbols

α	(5.6)
β	(5.8)
γ	(3.3)
ϕ	$= mn/\rho_p$
ϕ'	(3.12)
ϕ_m	value of ϕ at which N_m is maximum
μ	fluid viscosity
ν	$= \mu/\rho_f$
ξ	(5.13)
η	(3.8)
η'	(2.14)
η'', η'''	(3.6)
κ	(4.1)
κ_m	value of κ at which N is maximum
θ, ζ	(2.11)
θ_m	(5.32)

Suffixes

c	critical value
cp	value at close packing
h	value for a homogeneous bed
m	refers to a maximum
n	value for neutral stability
r, i	real and imaginary parts
0	value for an isolated particle
1	value at $\phi = \phi_1$

I am glad to acknowledge the considerable help with the preparation of this paper that I have received from Professor Roy Jackson of Princeton University.

REFERENCES

ANDERSON, T. B. & JACKSON, R. 1968 A fluid mechanical description of fluidized beds – stability of the state of uniform fluidization. *Ind. Engng Chem. Fundam.* **7**, 12–21.

BATCHELOR, G. K. 1972 Sedimentation in a dilute dispersion of spheres. *J. Fluid Mech.* **52**, 245–68.

BATCHELOR, G. K. 1976 Brownian diffusion of particles with hydrodynamic interaction. *J. Fluid Mech.* **74**, 1–29.

BIESHEUVEL, A. & VAN WIJNGAARDEN, L. 1984 Two-phase flow equations for a dispersion of gas bubbles in liquid. *J. Fluid Mech.* **148**, 301–318.

BUSCALL, R., GOODWIN, J. W., OTTEWILL, R. H. & TADROS, T. F. 1982 The settling of particles through Newtonian and non-Newtonian media. *J. Colloid Interface Sci.* **85**, 78–86.

CLIFT, R., GRACE, J. R. & WEBER, M. E. 1978 *Bubbles, Drops and Particles*. Academic.

DALLAVALLE, J. M. 1948 *Micromeritics*. Pitman.

DAVIDSON, J. F. & HARRISON, D. 1963 *Fluidized Particles*. Cambridge University Press.

DAVIS, R. H. & HASSAN, M. 1988 Spreading of the interface at the top of a slightly polydisperse sedimenting suspension. *J. Fluid Mech.* (in press).

DREW, D. A. 1983 Mathematical modeling of two-phase flow. *Ann. Rev. Fluid Mech.* **15**, 261–91.

EL-KAISSY, M. M. & HOMSY, G. M. 1976 Instability waves and the origin of bubbles in fluidized beds. Part I. Experiments. *Intl. J. Multiphase Flow* **2**, 379–395.

Fanucci, J. B., Ness, N. & Yen, R.-H. 1979 On the formation of bubbles in gas-particulate fluidized beds. *J. Fluid Mech.* **94**, 353–367.

Foscolo, P. U. & Gibilaro, L. G. 1984 A fully predictive criterion for the transition between particulate and aggregative fluidization. *Chem. Engng Sci.* **39**, 1667–75.

Foscolo, P. U. & Gibilaro, L. G. 1987 Fluid dynamic stability of fluidised suspensions: the particle bed model. *Chem. Engng Sci.* **42**, 1489–1500.

Garg, S. K. & Pritchett, J. W. 1975 Dynamics of gas-fluidized beds. *J. Appl. Phys.* **46**, 4493–4500.

Geldart, D. 1973 Types of gas fluidization. *Powder Tech.* **7**, 285–292.

Homsy, G. M., El-Kaissy, M. M. & Didwania, A. 1980 Instability waves and the origin of bubbles in fluidized beds. Part II. Comparison with theory. *Intl. J. Multiphase Flow* **6**, 305–318.

Jackson, R. 1963 The mechanics of fluidized beds. I. The stability of the state of uniform fluidization. *Trans. Inst. Chem. Engrs* **41**, 13–21.

Jackson, R. 1985 Hydrodynamic stability of fluid-particle systems. In *Fluidization* (ed. J. F. Davidson, R. Clift & D. Harrison), chap. 2. Academic.

Kynch, G. J. 1952 A theory of sedimentation. *Trans. Faraday Soc.* **48**, 166–76.

Molerus, O. 1967 Hydrodynamische Stabilitat des Fliessbetts. *Chem. Ing. Technik* **39**, 341–8.

Murray, J. D. 1965 On the mathematics of fluidization. I. *J. Fluid Mech.* **21**, 465–493.

Needham, D. J. & Merkin, J. H. 1983 The propagation of a voidage disturbance in a uniformly fluidized bed. *J. Fluid Mech.* **131**, 427–454.

Pigford, R. L. & Baron, T. 1965 Hydrodynamic stability of a fluidized bed. *Ind. Engng Chem. Fundam.* **1**, 81–87.

Ruckenstein, E. 1962 Uber die Stabilitat der Homogenen Struktur von Wirbelschichten. *Revue de Physique* **7**, 137–143.

Verloop, J. & Heertjes, P. M. 1970 Shock waves as a criterion for the transition from homogeneous to heterogeneous fluidization. *Chem. Engng Sci.* **25**, 825–832.

Wallis, G. B. 1962 One-dimensional waves in two-component flow. Atomic Energy Establishment, Winfrith, Report no. R162.

Wallis, G. B. 1969 *One-dimensional Two-phase Flow*. McGraw-Hill.

Wilhelm, R. H. & Kwauk, M. 1948 Fluidization of solid particles. *Chem. Engng Prog.* **44**, 201–218.

Zuber, N. 1964 On the dispersed two-phase flow in the laminar flow regime. *Chem. Engng Sci.* **19**, 897.